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МНОГОМЕРНАЯ МАТЕМАТИКА. МОДЕЛИРОВАНИЕ СТРУКТУРЫ МАТЕРИАЛА И ИНЖЕНЕРИЯ: ИНТЕРПРЕТАЦИЯ N-МЕРНОЙ СИСТЕМЫ В ДВУХМЕРНУЮ И ВЫБОР ОПТИМАЛЬНЫХ ПАРАМЕТРОВ
MULTIDIMENSIONAL MATHEMATICS. MATERIAL STRUCTURE MODELING AND ENGINEERING: INTERPRETING AN N-DIMENSIONAL SYSTEM INTO A TWO-DIMENSIONAL SYSTEM AND SELECTING THE OPTIMAL PARAMETERS

Аннотация. Цель работы состоит в компьютерной инженерии и материаловедение: моделирование, интерпретация N-мерной в двухмерную и выбор оптимальных параметров инженерной системы на базе многомерной математики (раздел: теории и методов векторной оптимизации). В рамках теории векторной оптимизации представлены принципы оптимальности решения векторных задач при равнозначных критериях и при заданном приоритете критерия https://rdcu.be/bhZ8i. (Работа "Vector optimization with equivalent and priority criteria" Springer Nature распространяется бесплатно.). На основе теории разработаны конструктивные методы решения задач векторной оптимизации, которые позволяют оценивать экспериментальные данные, во-первых, при равнозначных критериях, во-вторых, при моделировании инженерных систем с заданным приоритетом критерия при принятии оптимального решения. Практическая направленность показана при автоматизированном проектировании на базе векторной оптимизации инженерных систем, которым относятся структура материала. Для этой цели разработано программное обеспечение решения векторных задач нелинейного (ВЗНП) программирования. Программное обеспечение решения ВЗНП используются при цифровой трансформации принятия оптимальных решений в инженерных задачах. Численные примеры представлены цифровой трансформацией принятия оптимальных решений по структуре материала.

При принятии оптимальных решений в инженерных системах разработано: построение исходных данных (техническое задание) для моделирования структуры материала; преобразование математической модели структуры материала в условиях неопределенности в модель в условиях определенности; принятие оптимального решения (которое включает параметры и характеристики материала) с равнозначными критериями; принятие оптимального решения с заданным приоритетом критерия.

Abstract. The purpose of the work is in computer engineering and materials science: modeling, interpretation of N-dimensional into two-dimensional and selection of optimal parameters of an engineering system based on multidimensional mathematics (section: theory and methods of vector optimization). Within the framework of the theory of vector optimization, the principles of optimality of solving vector problems with equivalent criteria and with a given priority of the criterion are presented https://rdcu.be/bhZ8i. (The work "Vector optimization with equivalent and priority criteria» by Springer Nature is distributed free of charge.). On the basis of the theory, constructive methods for solving vector optimization problems have been developed, which make it possible to evaluate experimental data, firstly, with equivalent criteria, and secondly, when modeling engineering systems with a given criterion priority when making an optimal decision. Practical orientation is shown in computer-aided design based on vector optimization of engineering systems, which include the structure of the material. For this purpose, software for solving vector problems of nonlinear programming (VPNP) has been developed. VPNP solution software is used in the digital transformation of optimal decision-making in engineering problems. Numerical examples are presented by the digital transformation of optimal decision-making on the structure of the material. To make the optimal decision in engineering systems (on the example of the structure of the material), the following has been developed: the construction of initial data (technical specification) for modeling the structure of the material; transformation of a mathematical model of the structure of a material under conditions of uncertainty into a model under conditions of certainty; making an optimal decision (which includes parameters and characteristics of the material) with equivalent criteria; making an optimal decision with a given priority criterion.

Ключевые слова: Инженерная система, Теория многомерной математики, Аксиоматика векторной оптимизации, Методология моделирования, Математическое и программное обеспечение, сложная техническая система структура материала.

Keywords: Engineering System, Theory of Multidimensional Mathematics, Axiomatics of Vector Optimization, Modeling Methodology, Mathematical and Software, Complex Technical System Material Structure.

1. Introduction.

The study of the development of engineering systems and materials science, in particular, has shown that their development depends on a certain set of functional characteristics that must be taken into account at the design stage. Analysis of the functioning of engineering systems showed that improvement in one of the characteristics leads to the deterioration of other characteristics. To improve the functioning of the engineering system as a whole, it is necessary to improve all characteristics as a whole, [1, 2].

The mathematical model of such (engineering) systems is represented by multi-criteria optimization problems and, as a result, the solution of multi-criteria (vector) problems of mathematical programming is required. Research on this class of problems began more than a hundred years ago in Pareto V. [3]. Further research on multi-criteria optimization was carried out both at the theoretical level by foreign [4, 5, 6, 34-39] and Russian authors [13, 14, 15-33], and on solving practical problems first in the field of Economics [15, 16, 44], and then in the field of engineering systems [7-12, 16-33, 44].

The purpose of the work is to analyze, build a mathematical model of the structure of the material, study the processes of digital transformation of the development of engineering systems in conditions of certainty and uncertainty, the choice of optimal parameters of the structure of the material based on the theory and methods of vector optimization.

Within the framework of the theory of vector optimization, the principles of optimality of solving vector problems with equivalent criteria and with a given priority of the criterion are presented and constructive methods for solving vector optimization problems are shown. For modeling and digital transformation of engineering systems, vector problems of nonlinear programming were used, which were solved under conditions of certainty and uncertainty.

To achieve this goal, the work presents two areas of research: mathematical, software and the applied field. In the field of mathematical and software, the characteristic is presented, the analysis and study of vector optimization problems is carried out, [13, 14, 15-33, 44]. Within the framework of the theory of vector optimization, axiomatics and principles of optimality for solving vector problems with equivalent criteria and with a given priority of the criterion are presented.

On the basis of the principles of optimality, constructive methods for solving vector optimization problems have been developed, which make it possible to make an optimal decision, firstly, with equivalent criteria, and secondly, with a given priority of the criterion. In the study of the problem of vector optimization, a numerical solution of the vector problem of nonlinear (convex) programming with four homogeneous criteria.

In the applied part of the work, in organizational terms, the process of modeling and simulating the structure of the material is presented in the form of a methodology: "Methodology for choosing the optimal parameters of engineering systems under conditions of certainty and uncertainty". (Technical systems [16-32], technological processes [16, 22], materials [18, 44]),

The tasks that arise in the process of making an optimal decision on the selection of optimal parameters of complex engineering systems include three types sequentially. 1 type. Solution of a vector problem with equivalent criteria. The result obtained is the basis for further research of the system. In this case, the method of solving a vector problem with equivalent criteria is used.

If the result obtained satisfies the decision-maker (decision-maker - designer), then it is taken as a basis. If the solution does not satisfy the decision-maker, then we move on to the 2nd type (direct task) related to changing the parameters; or the 3rd type of solving vector problems (Inverse problem: "What will be the parameters of complex technical systems with given characteristics).

In organizational terms, the process of modeling and simulation of a complex engineering system, which includes three types of the above tasks, is formed in the form of a methodology: "*Methodology for Selecting the Optimal Parameters of Complex Engineering Systems in Conditions of Certainty and Uncertainty*", [18, 44].

The methodology includes three blocks, divided into a number of stages: Block 1. Formation of technical specifications, transformation of uncertainty conditions into certainty; Block 2. Methodology of the process of optimal decision making (selection of optimal parameters) in an engineering system based on vector optimization (the process of simulation of an engineering system). Block 3. Research, design, geometric interpretation of the transition from N-dimensional space and selection of optimal parameters of a complex engineering system (material structure) in multidimensional mathematics.

The implementation of all blocks of the methodology is shown by a numerical example (material structure).

2. Problem statement. Building a Mathematical Model of Material Structure under Conditions of Certainty and Uncertainty.

Chemical composition of material of a product is defined (on unit of volume, weights) by the percentage maintenance of some set of components of material which are equal in the sum to hundred percent. The composition of material, is characterized by a particular set of the functional characteristics which include mechanical and physical and chemical characteristics of materials. One group of properties (the functional characteristics) of material is characterized by the fact that it is desirable to receive them on the numerical value as much as possible (for example, durability), other group of properties is characterized by the fact that is desirable to receive them on the numerical value less as. Improvement on one of these characteristics leads another to deterioration. In general, it is required to pick up such composition of material that all properties of material were as it is possible better in total.

2.1. Mathematical model of structure of material

Discusses the composition of the material, any product, technical system that depends on a number of material component: $Y = \{y_1, y_2, \dots, y_V\}$, where $\boldsymbol{V}$ is the set of components of the material, $Y = \{y_j, j = \overline{1,V}\}$, V is the number of components of which it can be made (fabricated) material, y_V is the size as a percentage *v*th of a material component, each of which lies in the given limits:

$$y_v^{min} \le y_v \le y_v^{max},\ v = \overline{1,V}, \quad (2.1)$$

where y_v^{min}, y_v^{max}, $\forall v \in V$ are the lower and upper limits of the change in the vector of the material components.

$$\sum_{v=1}^{V} y_v(t) = 100\%, \quad (2.2)$$

the sum of all the components of the material is one hundred percent.

The composition of material is estimated by set K physical properties of material:

$$H(Y) = \{h_k(Y), k = \overline{1,K}\}, \quad (2.3)$$

which functionally depend on design data of $Y = \{y_j, j = \overline{1,V}\}^T$;

k is the index of a type of physical properties of material, $k = \overline{1,K}$, where K - number of types of properties (the functional characteristics) of material, we will present them in the form a vector – functions.

$H(Y)$ *is* a vector function (vector criterion) having K a component function:

$$H(Y) = \{h_k(Y), k = \overline{1,K}\}.$$

The set $\boldsymbol{K}$ consists of sets of $\boldsymbol{K}_1$, a component of maximization and $\boldsymbol{K}_2$ of minimization;

$\boldsymbol{K} = \boldsymbol{K}_1 \cup \boldsymbol{K}_2$;

$H_1(Y) = \{h_k(Y),\ k = \overline{1,K_1}\}$ is maximizing vector-criterion, K_1 – number of criteria, and $K_1 \equiv \overline{1,K_1}$ is a set of maximizing criteria. Let's further assume that $H_1(Y) = \{h_k(Y),\ k = \overline{1,K_1}\}$ is the continuous concave functions (we will sometimes call them the maximizing criteria);

$H_2(Y) = \{h_k(Y),\ k = \overline{1, K_2}\}$ is vector criterion in which each component is minimized, $K_2 \equiv \overline{K_1 + 1, K} \equiv \overline{1, K_2}$ - a set of minimization criteria, K_2 – number. We assume that $h_k(Y), k = \overline{1, K_2}$ is the continuous convex functions (we will sometimes call these the minimization criteria), i.e.:

$K_1 \cup K_2 = \boldsymbol{K}, K_1 \subset \boldsymbol{K}, \boldsymbol{K_2} \subset \boldsymbol{K}$.

We use characteristics of the material $H(Y) = \{h_k(Y), k = \overline{1, K}\}$

as criterion, and change limits imposed on each type of components as parametrical restrictions. We will present the mathematical model of material solving in general a problem of the choice of the optimal design solution (the choice of optimum structure of material) in the form of a vector problem of mathematical programming:

$$Opt\ H(Y) = \{\max H_1(Y) = \{\max h_k\ (Y),\ k = \overline{1, K_1}\ , \quad (2.4)$$

$$\min H_2(Y) = \{\min h_k\ (Y),\ k = \overline{1, K_2}\}\}, \quad (2.5)$$

at restrictions $G(X) \leq B$, (2.6)

$$\textstyle\sum_{v=1}^{V} y_v(t) = 100\%\ , \quad (2.7)$$

$$Y_v^{min} \leq y_v \leq Y_v^{max}, v = \overline{1, V}, \quad (2.8)$$

where $Y = \{y_j, j = \overline{1, V}\}$ is a vector of the operated variables (a material component) from (2.1);

$H(Y) = \{h_k(Y), k = \overline{1, K}\}$ is vector criterion which each function submits the characteristic (property) of material which is functionally depending on a vector of variables Y;

$G(Y) = \{g_1(Y), \ldots, g_M(Y)\}^T$ is a vector function of the restrictions imposed on structure of material, $\boldsymbol{M}$ – a set of restrictions.

It is supposed that the functions $H(Y) = \{h_k(Y), k = \overline{1, K}\}$ are differentiated and convex, $G(Y) = \{g_i(Y), i = \overline{1, M}\ \}^T$ are continuous, and (2.6)-(2.8) set of admissible points of $\boldsymbol{S}$ set by restrictions are not empty and represents a compact:

$$\boldsymbol{S} = \{X \in \boldsymbol{R}^n | G(X) \leq 0, X^{min} \leq X \leq X^{max}\} \neq \emptyset\ - \quad (2.9)$$

The relations (2.4)-(2.8) form mathematical model of material. It is required to find such vector of the $Y^o \in \boldsymbol{S}$ parameters at which each component (characteristic) the vector - functions $H_1(Y)$ accepts the greatest possible value, and a vector - functions $H_2(Y)$ accepts minimum value:

$$H_1(Y) = \{h_k(Y),\ k = \overline{1, K_1}\},$$

$$H_2(Y) = \{h_k(Y),\ k = \overline{1, K_2}\}. \quad (2.10)$$

In this article a research of design properties of material is considered in statics. However, structure of material can be considered in dynamics (for example, at change of external temperature for some period of time). For this purpose, it is possible, to use differential-difference methods of transformation [4] and to conduct research for a small discrete time term$\Delta t \in \boldsymbol{T}$. In set the mathematical model of material (2.4)-(2.8) can be treated as systems approach to a material research.

2.2. Creation of mathematical model of structure of material in the conditions of certainty and uncertainty

At creation of mathematical model of material (2.4)-(2.8), as well as for technical system [20-26], conditions are possible: certainty and uncertainty.

2.2.1. Creation of mathematical model of material in the conditions of certainty

Conditions of a certainty are characterized by the fact that the functional dependence of each characteristic (property) of material and restrictions on design components of material is known. For creation of the functional dependence, we perform the following works.

1. We form a set of all functional characteristics (properties) of material $\boldsymbol{K}$. The size of the characteristic we will designate $h_k(Y), k = \overline{1, K}$. We define a set of all components of material $\boldsymbol{V}$ on which these characteristics depend. We will present sizes of parameters in the form of a vector of $Y = \{y_j, j = \overline{1, V}\}$. We give the verbal description of characteristics of material.

2. We conduct research of the physical processes proceeding in material. For this purpose, we use fundamental laws of physics: model operation of magnetic, temperature profiles; laws of conservation of energy, movements etc. We establish informational and functional connection of characteristics of material and its parameters:

$H(Y) = \{h_k(Y), k = \overline{1, K}\}$.

3. We define the functional restrictions:

$h_k^{min} \leq h_k \leq h_k^{max}, k = \overline{1, K}$, or $H^{min} \leq H \leq H^{max}$;

and parametrical restrictions:

$y_v^{min} \le y_v \le y_v^{max}, v = \overline{1,V}$, or $Y^{min} \le Y \le Y^{max}$.

The sum of all components of material is equal to hundred percent: $\sum_{v=1}^{V} y_v(t) = 100\%$.

4. As a result we will construct mathematical model of material in the form of a vector problem of mathematical programming:

$$Opt\ H(Y) = \{\max H_1(Y) = \{\max h_k\ (Y),\ k = \overline{1,K_1}\}, \quad (2.10)$$

$$\min H_2(Y) = \{\min h_k\ (Y),\ k = \overline{1,K_2}\}\}, \quad (2.11)$$

$$H^{min} \le H \le H^{max}, \quad (2.12)$$

$$\text{at restrictions} \quad \sum_{v=1}^{V} y_v(t) = 100\%\ , \quad (2.13)$$

$$y_v^{min} \le y_v \le y_v^{max}, v = \overline{1,V}, \quad (2.14)$$

The problem (2.10)-(2.14) is adequate problem (2.4)-(2.8).

2.2.2. Creation of mathematical model of material in the conditions of uncertainty

Conditions of uncertainty are characterized by the fact that there is no sufficient information on the functional dependence of property of material from structure of components. In this case the pilot studies are conducted.

For the given number of compositions of the materials:

$Y_i = \{y_{iv}, v = \overline{1,V}\}, i = \overline{1,M}$,

the corresponding set of properties are defined:

$H(Y_i) = \{h_k(Y_i), k = \overline{1,K}\}, i = \overline{1,M}$.

Taking into account it the matrix of experiments on research of structure of material takes a form:

$$\mathrm{I} = \left\| \begin{matrix} Y_1 = \{y_{1v}, v = \overline{1,V_1}\}\ h_1(Y_1) \dots\ h_K(Y_1) \\ \dots \\ Y_M = \{y_{Mv}, v = \overline{1,V_M}\}\ h_1(Y_M) \dots\ h_K(Y_M) \end{matrix} \right\|. \quad (2.15)$$

where the $v \in \boldsymbol{V}$ column represents a numerical value vth of a material component as a percentage, $v = \overline{1,V}$, and the $k \in \boldsymbol{K}$ column represents a numerical value of kth of property of material, $k = \overline{1,K}$. The problem of the person, the making decision, (the designer) consists in the choice of such alternative which would allow to receive "in the greatest measure (optimum) result arranging it" [18, 20]. The set of criteria (characteristics) of $\boldsymbol{K}$ is subdivided into two subsets of $\boldsymbol{K} = \boldsymbol{K}_1 \cup \boldsymbol{K}_2, \boldsymbol{K}_1 \subset \boldsymbol{K}, \boldsymbol{K}_2 \subset \boldsymbol{K}$.

$\boldsymbol{K}_1$ is a subset of characteristics which numerical values it is desirable to receive as it is possible above:

$I_1(Y_i) = \{h_k(Y_i, i = \overline{1,M}), k = \overline{1,K_1}\} \to max$.

$\boldsymbol{K}_2$ are subsets of principal specifications which numerical values it is desirable to receive, as low as possible:

$$I_2(Y_i) = \{h_k(Y_i, i = \overline{1,M}), k = \overline{1,K_2}\} \to min.$$

The solution of a problem of a decision making on structure of material (2.15) it is in essence close to the solution of a vector problem of mathematical programming which in the conditions of uncertainty will take a form:

$$Opt\ H(Y) = \{\max I_1(Y) = \{\max h_k\ (Y_i, i = \overline{1,M}),\ k = \overline{1,K_1}\}, \quad (2.16)$$

$$\min I_2(Y) = \{\min h_k\ (Y_i, i = \overline{1,M}),\ k = \overline{1,K_2}\}\}, \quad (2.17)$$

$$\text{at restrictions}\ h_k^{min}(Y_i, i = \overline{1,M}) \le h_k \le h_k^{max}(Y_i, i = \overline{1,M}), k = \overline{1,K}, \quad (2.18)$$

$$\sum_{v=1}^{V} y_v(t) = 100\%\ , \quad (2.19)$$

$$y_v^{min} \le y_v \le y_v^{max}, v = \overline{1,V}, \quad (2.20)$$

where $Y_i = \{y_{ij}, j = \overline{1,V}\}, i = \overline{1,M}$ is a vector of operated variable (constructive parameters);

$H(Y_i) = \{I_1(Y_i), I_1(Y_i)\}$ is vector criterion which each component submits the characteristic (property) of material which is functionally depending on the size of discrete value of a vector of variables $Y_i, i = \overline{1,M}$; M is set of discrete values of a vector of the variables $Y_i, i = \overline{1,M}$;

in (2.18) $h_k^{min}(Y_i) \le h_k \le h_k^{max}(Y_i), k = \overline{1,K}$ is a vector function of the restrictions imposed on function of material of a product, $y_v^{min} \le y_v \le y_v^{max}, v = \overline{1,V}$ are parametrical restrictions.

2.3. Creation of mathematical model of material in the conditions of certainty and uncertainty in the form of a vector problem

In actual life of a condition of a certainty and uncertainty are combined. The material model also reflects these conditions. Let's unite models (2.10)-(2.14) and (2.16)-(2.20). As a result, we will receive material model in the conditions of certainty and uncertainty in total in the form of a vector problem of mathematical programming:

$$Opt\ H(Y) = \{\max H_1(Y) = \{\max h_k\,(Y),\ k = \overline{1, K_1^{def}}\}, \tag{2.21}$$

$$\max I_1(Y) = \{\max h_k\,(Y_i, i = \overline{1, M}),\ k = \overline{1, K_1^{unc}}\}, \tag{2.22}$$

$$\min H_2(Y) = \{\min h_k\,(Y),\ k = \overline{1, K_2^{def}}\}, \tag{2.23}$$

$$\min I_2(Y) = \{\min h_k\,(Y_i, i = \overline{1, M}),\ k = \overline{1, K_2^{unc}}\}\}, \tag{2.24}$$

$$\text{at restrictions}\quad h_k^{min} \le h_k \le h_k^{max}, k = \overline{1, K}, \tag{2.25}$$

$$\sum_{v=1}^{V} y_v = 100\%\ , \tag{2.26}$$

$$y_v^{min} \le y_v \le y_v^{max}, v = \overline{1, V}, \tag{2.27}$$

where Y - a vector of the operated variables (design data of material);

$H(Y) = \{H_1(Y)\ I_1(Y)\ H_2(Y)\ I_2(Y)\}$ is vector criterion which each component represents a vector of criteria (characteristics) of material which functionally depend on values of a vector of variables Y; K_1^{def}, K_2^{def} (*definiteness*), K_1^{unc}, K_2^{unc} (*uncertainty*) - the set of criteria of max and min created in the conditions of a certainty and uncertainty; in (2.25) vector function of the restrictions imposed on material functioning under production conditions, (2.27) parametrical restrictions.

3. Introduction to Multidimensional Mathematics: Analysis, Vector Problem of Mathematical Programming, Theory, Axioms and Axiomatic Methods, Principles of Optimality

Mathematical models of the structure of material (2.21)-(2.27), as well as models of technical systems, technological processes and dynamical systems are represented by vector problems of mathematical programming (VPMP), [16 - 21, 44]. Further development of the study of works on the theory of vector optimization led to the formation of "Multidimensional Mathematics".

3.1. Analysis of the Development of Modern Mathematics.

The analysis of modern mathematics was carried out in accordance with [1, pp. 560 – 563].

Mathematics is the science of quantitative relations and spatial forms of the real world. Mathematics, as a science, became possible after the accumulation of sufficiently large factual material, arose in ancient Greece in the 6th – 5th centuries BC, in accordance with [1] four periods.

1. The origin of mathematics. In the early stages of development, counting objects of existence led to the creation of the simplest concepts of arithmetic of natural numbers.

2. The period of elementary mathematics. The study of the objects of existence led to the creation of the simplest concepts of arithmetic calculations, the determination of areas, volumes, etc.

3. The period of creation of the mathematics of variables. In the 17th century, a new period in the development of mathematics began. ***The concept of function***, which determines the interrelation of variables (parameters) of the object under study, comes to the fore. The study of variables and functional dependencies leads further to the basic concepts of mathematical analysis, to the concept of limit, derivative, differential, and integral. An analysis of infinitesimals is created in the form of differential and integral calculations, which makes it possible to relate finite changes in variables to their behavior on the decision (function) being made. The basic laws of mechanics and physics are written in the form of differential equations, and the task of integrating these equations is one of the most important tasks of mathematics.

4. Modern mathematics. All of the branches of mathematical analysis created in the 17th and 18th centuries continued to evolve into the 19th, 20th, and 21st centuries. As the basic apparatus of the new fields of mechanics and mathematical physics, the theory of ordinary differential equations, partial differential equations, and computational mathematics is being intensively developed. The problems of finding the best solution in the problems of controlling physical or mechanical systems, described by differential equations, led to the creation of the theory of optimal control.

In general, the process of development of mathematics shows that when solving mathematical problems, there was a study and analysis of a separate ***function (one-dimensional)***, depending on a certain set of variables (***parameters***) of the object or system under study. (For more details, see [1, pp. 560 – 563]).

In real life, the object under study, the system, in its functioning (development), is characterized by a certain ***set of functional characteristics*** that depend on the same parameters of the system. Hence, the problem of multidimensionality of the objects and systems under study has become a general scientific one.

To solve the problem of multidimensionality, we will present a vector (multidimensional) optimization problem and consider the theory (axiomatics, principles of optimality) of its solution, [15, 29, 44].

3.2. A vector problem of mathematical programming

A vector problem in mathematical programming (VPMP) is a standard mathematical-programming problem including a set of criteria, which, in total, represent a vector of criteria.

It is important to distinguish between uniform and non-uniform VPMP:

A uniform maximizing VPMP is a vector problem in which each criterion is directed towards maximizing;

A uniform minimizing VPMP is a vector problem in which each criterion is directed towards minimizing;

A non-uniform VPMP is a vector problem in which the set of criteria is shared between two subsets (vectors) of criteria (maximization and minimization respectively), e.g., non-uniform VPMP are associated with two types of uniform problems.

According to these definitions, we will present a vector problem in mathematical programming with non-uniform criteria [6, 20, 22] in the following form:

$$Opt\, F(X) = \{\max F_1(X) = \{\max f_k\,(X),\ k = \overline{1, K_1}, \tag{3.1}$$

$$\min F_2(X) = \{\min f_k\,(X),\ k = \overline{1, K_2}\}\}, \tag{3.2}$$

$$G(X) \leq B, \tag{3.3}$$

$$X \geq 0, \tag{3.4}$$

where $X = \{x_j,\ j = \overline{1, N}\}$ is a vector of material variables, N-dimensional Euclidean space of R^N, (designation $j = j = \overline{1, N}$ is equivalent to $j = 1,...,N$);

$F(X)$ *is* a vector function (vector criterion) having K – a component functions, (K - set power $\boldsymbol{K}$), $F(X) = \{f_k(X),\ k = \overline{1, K}\}$. The set $\boldsymbol{K}$ consists of sets of $\boldsymbol{K}_1$, a component of maximization and $\boldsymbol{K}_2$ of minimization; $\boldsymbol{K}=\boldsymbol{K}_1\cup\boldsymbol{K}_2$ therefore we enter the designation of the operation "*opt*," which includes *max* and *min*;

$F_1(X) = \{f_k(X),\ k = \overline{1, K_1}\}$ is maximizing vector-criterion, K_1 – number of criteria, and $K_1 \equiv \overline{1, K_1}$ is a set of maximizing criteria (a problem (3.1), (3.3), (3.4) represents VPMP with the homogeneous maximizing criteria). Let's further assume that $f_k(X), k = \overline{1, K_1}$ is the continuous concave functions (we will sometimes call them the maximizing criteria);

$F_2(X) = \{f_k(X),\ k = \overline{1, K_2}\}$ is vector criterion in which each component is minimized, $K_2 \equiv \overline{K_1 + 1, K} \equiv \overline{1, K_2}$ - a set of minimization criteria, $\boldsymbol{K}_2$ – number, (the problems (3.2)-(3.4) are VPMP with the homogeneous minimization criteria). We assume that $f_k(X), k = \overline{1, K_2}$ is the continuous convex functions (we will sometimes call these the minimization criteria), i.e.,

$$\boldsymbol{K}_1\cup\boldsymbol{K}_2 = \boldsymbol{K},\ \boldsymbol{K}_1\subset\boldsymbol{K},\ \boldsymbol{K}_2\subset\boldsymbol{K}. \tag{3.5}$$

$G(X) \leq B, X \geq 0$ is standard restrictions, $g_i(X) \leq b_i, i = 1, \dots, M$ where b_i - a set of real numbers, and $g_i(X)$ are assumed continuous and convex.

$$\boldsymbol{S} = \{X \in \boldsymbol{R}^n | X \geq 0, G(X) \leq B, X^{min} \leq X \leq X^{max}\} \neq \emptyset, \tag{3.6}$$

where the set of admissible points set by restrictions (3.3)-(3.4) is not empty and represents a compact. The vector minimization function (criterion) $F_2(X)$can be transformed to the vector maximization function (criterion) by the multiplication of each component of $F_2(X)$ to minus unit. The vector criterion of $F_2(X)$ is injected into VPMP (3.1)-(3.4) to show that, in a problem, there are two subsets of criteria of $\boldsymbol{K}_1$, $\boldsymbol{K}_2$ with, in essence, various directions of optimization.

We assume that the optimum points received by each criterion do not coincide for at least two criteria. If all points of an optimum coincide among themselves for all criteria, then we consider the decision trivially.

3.3. The theory of vector optimization

The theory of vector optimization is aimed at solving vector problems of mathematical programming (3.1) - (3.4) with uniform and non-uniform criteria. The theory of vector optimization includes theoretical foundations: axiomatics, principles of optimality, and methods for solving vector problems, firstly, with equivalent criteria and, secondly, with a given priority of the criterion.

In accordance with this definition, the "Theory of Vector Optimization" includes the following sections. Basic theoretical concepts and definitions that will be used in the construction of axiomatics (axiomatics of Y.K. Mashunin), principles of optimality and methods for solving problems of vector optimization. The axiomatics of Y.K. Mashunin is divided into axiomatics, principles of optimality and methods for solving vector problems, firstly, with equivalent criteria and secondly, with a given priority of the criterion.

The concept of solving vector optimization problems with equivalent criteria. The concept of vector optimization with criterion priority. Symmetry in Vector Problems of Mathematical Programming: Research, Analysis.

Collectively, the theory of vector optimization represents the mathematical apparatus of modeling and making the optimal decision of the "object of decision-making".

The "object of decision-making" is: the social system, the economic and technical system. The mathematical apparatus allows you to choose any point from the set of points optimal according to Pareto, and show its optimality. We presented axiomatics, the principle of optimality and methods for solving problems of vector optimization (3.1) - (3.4) with equivalent criteria and a given priority of criteria. [6, 20]. For simplicity of research, the criteria and limitations of VPMP (3.1) - (3.4) are represented by polynomials of the second degree, i.e. convex vector problems are considered, which also include vector linear programming problems. Convex VPMP are characterized by the property that an optimum point exists and there is only one such point (Weierstrass Theorem).

3.4. Theoretical Foundations: Axioms and Axiomatic Methods.

An axiom is a statement that does not require logical proof. On the basis of these statements (initial assumptions), one or another theory is built.

The axiomatic method is a method of constructing a scientific theory, in which the theory is based on some initial assumptions called the axioms of the theory. As a result, all other provisions of the theory are obtained as logical consequences of axioms [41, 2].

In mathematics, the axiomatic method originated in the works of ancient Greek geometers. An example of the axiomatic method is the ancient Greek scientist Euclid, whose axioms were laid down in his famous work "Elements".

The axiomatic method was further developed in the works of D. Hilbert in the form of the so-called method of system formalism. The general scheme of building an arbitrary formal system ("S") includes:

1. *The language of the system* ("*S*"), including the alphabet – this is a list of elementary symbols; the rules of formation (syntax) on which the formulas "S" are built.

2. *Axioms of the* "*S*" system, which represent a certain set of formulas.

3. *Rules* for the withdrawal of the "*S*" system [41].

In the application to the solution of the problem of vector optimization (multidimensional mathematics), axiomatics is divided into two sections: 1. Axiomatics of solving the vector optimization problem with equivalent criteria; 2. Axiomatics of solving the vector optimization problem with a given priority of criteria. Only with the construction of the initial axiomatics is it possible in the future to construct the principle of optimality and the resulting algorithms for solving vector problems of mathematical programming.

4. Theory, axiomatics, the principle of optimality and methods for solving vector optimization problems: equivalent criteria and with a Criterion Priority

4.1 ***Theory, axiomatics, the principle of optimality and methods for solving vector optimization problems: equivalent criteria***

The axiomatics of vector optimization with equivalent criteria, as well as the theoretical axiomatics recommended by D. Hilbert [41, p. 111], includes three sections: 1) the language of the system in the form of definitions of the normalization of criteria and relative evaluation; 2) the axiomatics of the equality of criteria in the vector optimization problem; 3) the principle of optimality of the solution of the vector problem, on the basis of which a constructive method for solving the vector optimization problem with equivalent criteria is formed.

4.1.1. System language: Normalization of criteria, relative assessment

Definition 1. ***Normalizing of the criterion***.

Normalizing criteria (mathematical operation: the shift plus rationing) presents a unique display of the function $f_k(X)\ \forall k \in K$, in a one-dimensional space of $\boldsymbol{R}^1$(the function $f_k(X)\ \forall k \in K$ represents a function of transformation from a *N*-dimensional Euclidean space of $\boldsymbol{R^N}$ in $\boldsymbol{R^1}$). To normalize criteria in vector problems, linear transformations will be used:

$$f_k(X) = a_k f'_k(X) + c_k \forall k \in K \text{ , or}$$

$$f_k(X) = (f'_k(X) + c_k)/a_k \ \ \forall k \in K, \qquad (4.1)$$

where $f'_k(X)\ , k = \overline{1,K}$ - aged (before normalization) value of criterion; $f_k(X), k = \overline{1,K}$ - the normalized value, a_k, c_k - constants.

Normalization of criteria (4.1) $f_k(X) = (f'_k(X) + c_k)/a_k\ \forall k \in K$ is a simple (linear) invariant transformation of a polynomial, as a result of which the structure of the polynomial remains unchanged. In the optimization problem, the normalization of criteria $f_k(X) = (f'_k(X) + c_k)/a_k\ \forall k \in K$ does not affect the result of the solution. Indeed, if the convex optimization problem is solved:

$$\boldsymbol{max}_{X\in S}\ f(X), \text{ then at the optimum point } X^* \in S\text{: } \frac{df(X^*)}{dX} = 0. \qquad (4.2)$$

In the general case (including the normalization of the criterion (1)), the problem is solved:

$$\boldsymbol{max}_{X\in S}\ (a_k f'_k(X) + c_k), \qquad (4.3)$$

then at the optimum point $X^* \in S$:

$$\frac{d(a_k f(X^*) + c_k)}{dX} = a_k \frac{d(f(X^*))}{dX} + \frac{d(c_k)}{dX} = 0. \qquad (4.4)$$

The result is identical, i.e. the optimum point $X^*_k, k = \overline{1,K}$ is the same for non-normalized and normalized problems.

Definition 2. ***Relative evaluation of the function (criterion)***.

In the vector problem (3.1)-(3.4), normalize (4.1) of the form:

$$\lambda_k(X) = \frac{f_k(X) - f^0_k}{f^*_k - f^0_k}, \forall k \in \boldsymbol{K}, \qquad (4.5)$$

$\lambda_k(X)$ is the relative estimate of a point $X \in \boldsymbol{S}$ *k*th criterion $f_k(X)$ - *k*th criterion at the point $X \in \boldsymbol{S}$;

f^*_k - value of the *k*th criterion at the point of optimum X^*_k, obtained in vector problem (3.1) - (3.4) of individual *k*th criterion; f^0_k is the worst value of the *k*th criterion (ant optimum) at the point X^0_k (Superscript 0 - zero) on the admissible set $\boldsymbol{S}$ in vector problem (3.1)-(3.4);

the task at max (3.1), (3.3), (3.4) the value of f^0_k is the lowest value of the *k*th criterion

$f^0_k = min_{X\in S} f_k(X)\ \forall k \in K_1$,

and task min (3.2), (3.3), (3.4) the value of f^0_k is the greatest value of the *k*th criterion:

$f^0_k = max_{X\in S} f_k(X)\ \forall k \in K_2$.

The relative estimate of the $\lambda_k(X)\ \forall k \in K$ is first, measured in relative units;

secondly, the relative assessment of the $\lambda_k(X)\ \forall k \in K$: on the admissible set is changed from zero in a point of X^0_k: $\forall k \in \boldsymbol{K}\ \lim_{X \to X^0_k} \lambda_k(X) = 0$, to the unit at the point of an optimum of X^*_k:

$\forall k \in \boldsymbol{K}\ \lim_{X \to X^*_k} \lambda_k(X) = 1$:

$$\forall k \in \boldsymbol{K}\ 0 \le \lambda_k(X) \le 1, X \in \boldsymbol{S}. \qquad (4.6)$$

As a result of this normalization, all the criteria of the VPMP are (3.1)-(3.4) are comparable in relative units, which allows comparing them with each other, using criteria for joint optimization.

Definition 3. ***The operation of comparing relative estimates of a function (criterion) with each other***.

Since any function (criterion) is represented in the relative estimates of the functions $\lambda_k(X)\ \forall k \in K$, which lie within the range of (4.6) $\forall k \in \boldsymbol{K}\ 0 \le \lambda_k(X) \le 1$, it is possible to compare the relative estimates by numerical value. For comparison, the "subtraction" operation is used. If two functions (criteria) measured in the relative estimates $\lambda_{k=1}(X)$ and $\lambda_{k=2}(X)\ \forall k \in K$ are compared, then three situations are possible:

the first, when $\lambda_{k=1}(X) > \lambda_{k=2}(X)$;

the second, when $\lambda_{k=1}(X) = \lambda_{k=2}(X)$; (4.7)

the third, when $\lambda_{k=1}(X) < \lambda_{k=2}(X)$. (4.8)

The first and third situations are explored in Section 6.

This section 5 examines the second situation.

4.1.2. Axiomatics of Vector Optimization with Equivalent Criteria

Axiom 1. ***On the equivalence of criteria at an admissible point of a vector problem of mathematical programming***.

In of vector problems of mathematical programming two criteria with the indexes $k \in \boldsymbol{K}, q \in \boldsymbol{K}$ shall be considered equivalent in $X \in \boldsymbol{S}$ point if relative estimates on kth and qth to criterion are equal among themselves in this point, i.e. $\lambda_k(X) = \lambda_q(X), k, q \in \boldsymbol{K}$.

Explanation. If at point $X \in \boldsymbol{S}$ the functions (criteria) are equal to:

$\lambda_l(X) = 0{,}45\ l \in \boldsymbol{K}$ and $\lambda_q(X) = 0{,}45, q \in \boldsymbol{K}$ (i.e., 45% of its optimal value, which in relative units is equal to 1), then such criteria are not "equal" to each other, but are equivalent in their numerical value. And each of them carries its own functional meaning, which can be obtained using the normalization of criteria (4.5).

Definition 4. ***Definition of a minimum level among all relative estimates of criteria.***

The relative level λ in a vector problem represents the lower assessment of a point of X∈S among all relative estimates of $\lambda_k(X), k = \overline{1,K}$:

$$\forall X \in \boldsymbol{S}\ \lambda \le \lambda_k(X), k = \overline{1,K}, \quad (4.9)$$

the lower level for performance of a condition (4.9) in an admissible point of $X \in \boldsymbol{S}$ is defined by a formula:

$$\forall X \in S\ \lambda = \boldsymbol{min}_{k \in K} \lambda_k(X). \quad (4.10)$$

Ratios (4.9) and (4.10) are interconnected. They serve as transition from operation (4.10) of definition of min to restrictions (4.9) and vice versa.

The level λ allows to unite all criteria in a vector problem one numerical characteristic of λ and to make over her certain operations, thereby, carrying out these operations over all criteria measured in relative units. The level λ functionally depends on the X ∈ $\boldsymbol{S}$ variable, changing X.

We can change the lower level - λ. From here we will formulate the rule of search of the optimum decision. Therefore, by changing X, we can change everything $\lambda_k(X), k = \overline{1,K}$ and, accordingly, the lower level $\lambda = \boldsymbol{min}_{k \in K} \lambda_k(X)$, which is a characteristic of a multidimensional (multifunctional) system.

Explanation. The value of the relative estimate $\forall k \in K\ \lambda_k(X)$ is a characteristic of a one-dimensional system, and the value of the minimum relative level $\lambda = \boldsymbol{min}_{k \in K} \lambda_k(X)$ is *a characteristic of multidimensional mathematics*.

4.1.3. The principle of optimality of solving a multidimensional (vector) optimization problem with equivalent criteria.

Definition 5. ***The principle of an optimality of solving a multidimensional (vector) optimization problem with equivalent criteria.***

The vector problem of mathematical programming at equivalent criteria is solved, if the point of $X^o \in \boldsymbol{S}$ and a maximum level of λ^o (the top index o - optimum) among all relative estimates such that is found

$$\lambda^o = \boldsymbol{max}_{X \in S} \boldsymbol{min}_{k \in K} \lambda_k(X) \quad (4.11)$$

Using interrelation of expressions (4.9) and (4.10), we will transform a maximin problem (4.11) to an extreme problem:

$$\lambda^o = \boldsymbol{max}_{X \in S}\ \lambda \quad (4.12)$$

$$\text{at restriction } \lambda \le \lambda_k(X), k = \overline{1,K}. \quad (4.13)$$

λ-problem (4.12)-(4.13) has (N+1) dimension, as a consequence of the result of the solution of λ-problem (4.12)-(4.13) represents an optimum vector of $\boldsymbol{X^o} \in R^{N+1}, (N+1)$ which component an essence of the value of the λ^o, i.e. $\boldsymbol{X^o} = \{x_1^o, x_2^o, \dots, x_N^o, x_{N+1}^o\}$, thus $x_{N+1}^o = \lambda^o$, and (N+1) a component of a vector of $\boldsymbol{X^o}$ selected in view of its specificity.

The received a pair of $\{\lambda^o, X^o\} = \boldsymbol{X^o}$ characterizes the optimum solution of λ-problem (4.12)-(4.13) and according to vector problem of mathematical programming (3.1)-(3.4) with the equivalent criteria, solved on the basis of normalization of criteria and the principle of the guaranteed result. We will call in the optimum solution of $\boldsymbol{X^o} = \{\lambda^o, X^o\}$, X^o - an optimal point, and λ^o - a maximum level. An important result of the algorithm for solving vector problems (3.1)-(3.4) with equivalent criteria is the following theorem.

Theorem 1. ***The theorem of the two contradictory criteria in the vector problem of mathematical programming with equivalent criteria***.

In convex vector problems of mathematical programming (3.1)-(3.4) at the equivalent criteria which is solved on the basis of normalization of criteria and the principle of the guaranteed result, in an optimum point of $\boldsymbol{X^o} = \{\lambda^o, X^o\}$ two criteria are always - denote their indexes $q \in \boldsymbol{K}, p \in \boldsymbol{K}$ (which in a sense are the most contradiction of the criteria $k = \overline{1,K}$), for which equality is carried out:

$$\lambda^o = \lambda_q(X^o) = \lambda_p(X^o), q, p \in \boldsymbol{K}, X \in \boldsymbol{S}, \tag{4.14}$$

and other criteria are defined by inequalities:

$$\lambda^o \le \lambda_k(X^o), \forall k \in \boldsymbol{K},\ q \ne p \ne k. \tag{4.15}$$

For the first time, the proof of Theorem 1 is presented in [15, p. 22], and later it is repeated in [9, p.234]. Along with the fact that the point X^o is the optimal solution of the VPMP.

4.1.4. A constructive method for solving the vector optimization problem with equivalent criteria.

To solve of the vector problems of mathematical programming (3.1)-(3.4) the methods based on axiomatic of the normalization of criteria and the principle of the guaranteed result, which follow from Axiom 1 and the principle of optimality 1. The constructive method for solving a vector optimization problem with equivalent criteria includes two blocks: the 1st block "System Analysis" is divided into three steps; 2nd block "Optimal decision-making", which includes two steps: construction of the problem and its solution.

Block 1. System analysis.

Step 1. The problem (3.1)-(3.4) by each criterion separately is solved, i.e. for $\forall k \in \boldsymbol{K}_1$ is solved at the maximum, and for $\forall k \in \boldsymbol{K}_2$ is solved at a minimum. As a result of the decision, we will receive: X_k^* - an optimum point by the corresponding criterion, $k = \overline{1,K}$; $f_k^* = f_k(X_k^*)$ – the criterion size *k*th in this point, $k = \overline{1,K}$.

Step 2. We define the worst value of each criterion on $\boldsymbol{S}$: f_k^0, $k = \overline{1,K}$. For what the problem (3.1)-(3.4) for each criterion of $k = \overline{1,K_1}$ on a minimum is solved:

$f_k^0 = \min f_k(X), G(X) \le B, X \ge 0, k = \overline{1,K_1}$.

The problem (3.1)-(3.4) for each criterion $k = \overline{1,K_2}$ maximum is solved:

$$f_k^0 = \max f_k(X), G(X) \le B, X \ge 0, k = \overline{1,K_2}.$$

As a result of the decision, we will receive: $X_k^0 = \{x_j, j = \overline{1,N}\}$ - an optimum point by the corresponding criterion, $k = \overline{1,K}$;

$f_k^0 = f_k(X_k^0)$ – the criterion size *k*th a point, $X_k^0, k = \overline{1,K}$.

Step 3. The system analysis of a set of points, optimum across Pareto, for this purpose in optimum points of $X^* = \{X_k^*, k = \overline{1,K}\}$, are defined sizes of criterion functions of $F(X^*)$ and relative estimates $\lambda(X^*)$, $\lambda_k(X) = \frac{f_k(X) - f_k^0}{f_k^* - f_k^0}, \forall k \in \boldsymbol{K}$:

$$F(X^*) = \{f_k(X_k^*), q = \overline{1,K}, k = \overline{1,K}\} = \begin{vmatrix} f_1(X_1^*), \dots, f_K(X_1^*) \\ \dots \\ f_1(X_K^*), \dots, f_K(X_K^*) \end{vmatrix}, \tag{4.16}$$

$$\lambda(X^*) = \{\lambda_q(X_k^*), q = \overline{1,K}, k = \overline{1,K}\} = \begin{vmatrix} \lambda_1(X_1^*), \dots, \lambda_K(X_1^*) \\ \dots \\ \lambda_1(X_K^*), \dots, \lambda_K(X_K^*) \end{vmatrix}. \tag{4.17}$$

Any relative score (4.17) lies within the range of $0 \le \lambda_k(X) \le 1, k = \overline{1,K}$.
From the results of the system analysis (4.16)-(4.17) the problem arises: To find such an (optimal) point at which all relative estimates are: $\lambda_q(X), q = \overline{1,K}$ were close to unity. To solve this problem is aimed λ-problem.

Block 2. Making the optimal decision in the VPMP. It includes two steps - 4, 5.

Step 4. Creation of the λ-problem.

Creation of λ-problem is carried out in two stages:

initially built the maximine problem of optimization with the normalized criteria which at the second stage will be transformed to the standard problem of mathematical programming called λ-problem.

For construction maximine a problem of optimization we use definition 2 - relative level:

$\forall X \in \boldsymbol{S} \quad \lambda = \boldsymbol{min}_{\boldsymbol{k \in K}} \lambda_k(X)$.

The bottom λ level is maximized on $X \in S$, as a result we will receive a maximine problem of optimization with the normalized criteria.

$$\lambda^o = \boldsymbol{max}_{\boldsymbol{X \in S}} \boldsymbol{min}_{\boldsymbol{k \in K}} \lambda_k(X). \tag{4.18}$$

At the second stage we will transform a problem (4.18) to a standard problem of mathematical programming:

$$\lambda^o = \boldsymbol{max}_{\boldsymbol{X \in S}}\, \lambda, \qquad \rightarrow \qquad \lambda^o = \boldsymbol{max}_{\boldsymbol{X \in S}}\, \lambda, \tag{4.19}$$

$$\lambda - \lambda_k(X) \le 0, k = \overline{1,K}, \quad \rightarrow \quad \lambda - \frac{f_k(X) - f_k^0}{f_k^* - f_k^0} \le 0, k = \overline{1,K}, \tag{4.20}$$

$$G(X) \le B, X \ge 0, \qquad \rightarrow \qquad G(X) \le B, X \ge 0, \tag{4.21}$$

where the vector of unknown of X has dimension of $N + 1$: $X = \{\lambda, x_1, \dots, x_N\}$.

Step 5. Solution of λ-problem.

λ-problem (4.19)-(4.21) is a standard problem of convex programming and for its decision standard methods are used. As a result of the solution of λ-problem it is received:

$\boldsymbol{X^o} = \{\boldsymbol{X^o}, \boldsymbol{\lambda^o}\}$ - an optimum point; (4.22)

$f_k(X^o), k = \overline{1,K}$ are values of the criteria in this point; (4.23)

$\lambda_k(X^o) = \frac{f_k(X) - f_k^0}{f_k^* - f_k^0}, k = \overline{1,K}$ are sizes of relative estimates; (4.24)

λ^o - the maximum relative estimates which is the maximum bottom level for all relative estimates of $\lambda_k(X^o)$, or the guaranteed result in relative units. λ^o guarantees that all relative estimates of $\lambda_k(X^o)$ more or are equal λ^o :

$$\lambda_k(X^o) \ge \boldsymbol{\lambda^o}, k = \overline{1,K} \text{ or } \boldsymbol{\lambda^o} \le \lambda_k(X^o), k = \overline{1,K}, X^o \in \boldsymbol{S}, \tag{4.25}$$

and according to the theorem 1 point of $X^o = \{\boldsymbol{\lambda^o}, x_1, \dots, x_N\}$ is optimum across Pareto.

4.2. Theory, axiomatics, the principle of optimality and methods for solving vector optimization problems: with a Criterion Priority

Definition 3 states that if we compare two functions (criteria) measured in relative estimates $\lambda_{k=1}(X)$ and $\lambda_{k=2}(X)$ $\forall k \in K$, then three situations are possible. The second situation, when $\lambda_{k=1}(X) = \lambda_{k=2}(X)$ is investigated in Section 3.2 (equivalent criteria). Situations: the first, when $\lambda_{k=1}(X) > \lambda_{k=2}(X)$, and the third, when $\lambda_{k=1}(X) < \lambda_{k=2}(X)$, are explored in this section. Such situations are defined as tasks with the priority of the criterion.

For development of methods of the solution of problems of vector optimization with a priority of criterion we use definitions as follows:

Priority of one criterion of vector problems, with a criterion priority over other criteria;

Numerical expression of a priority;

The set priority of a criterion;

the lower (minimum) level from all criteria with a priority of one of them; a subset of points with priority by criterion (Axiom 2); the principle of optimality of the solution of problems of vector optimization with the set priority of one of the criteria, and related theorems. For more details see [17, 43, 44].

4.2.1. Axiomatics of solving a Vector Optimization Problem with a given criterion priority

The language of the axiomatics system for solving a vector problem with a given criterion priority includes definitions: 1) Priority of one criterion over another; 2) The numerical value of the priority of the criterion; 3) The lowest level of the criterion among all relative evaluations with the priority of the criterion.

*Definition 6. **About the priority of one criterion over the other***.

The criterion of $q \in \boldsymbol{K}$ in the vector problem of Equations (3.1)-(3.4) in a point of $X \in \boldsymbol{S}$ has priority over other criteria of $k = \overline{1, K}$, and the relative estimate of $\lambda_q(X)$ by this criterion is greater than or equal to relative estimates of $\lambda_k(X)$ of other criteria, i.e.:

$$\lambda_q(X) \geq \lambda_k(X), k = \overline{1, K}, \tag{4.26}$$

and a strict priority for at least one criterion of $t \in \boldsymbol{K}$,

$$\lambda_q(X) > \lambda_k(X), t \neq q, \text{ and for other criteria of } \lambda_q(X) \geq \lambda_k(X), k = \overline{1, K}, k \neq t \neq q. \tag{4.27}$$

Introduction of the definition of a priority of criterion $q \in \boldsymbol{K}$ in the vector problem of Equations (3.1)-(3.4) executed the redefinition of the early concept of a priority. Earlier the intuitive concept of the importance of this criterion was outlined, now this "importance" is defined as a mathematical concept: the higher the relative estimate of the qth criterion compared to others, the more it is important (i.e., more priority), and the highest priority at a point of an optimum is $X_k^*, \forall q \in \boldsymbol{K}$.

From the definition of a priority of criterion of $q \in \boldsymbol{K}$ in the vector problem of Equations (3.1)-(3.4), it follows that it is possible to reveal a set of points $\boldsymbol{S}_q \subset \boldsymbol{S}$ that is characterized by

$\lambda_q(X) \geq \lambda_k(X), \forall k \neq q, \forall X \in \boldsymbol{S}_q$. However, the answer to whether a criterion of $q \in \boldsymbol{K}$ at a point of the set $\boldsymbol{S}_q$ has more priority than others do remains open. For clarification of this question, we define a communication coefficient between a couple of relative estimates of q and k that, in total, represent a vector:

$$P^q(X) = \{p_k^q(X) \mid k = \overline{1, K}\}, q \in \boldsymbol{K}\ \forall X \in \boldsymbol{S}_q. \tag{4.28}$$

Definition 7. ***About numerical expression of a priority of one criterion over another***.

In the vector problem of Equations (3.1) and (3.4), with priority of the qth criterion over other criteria of $k = \overline{1, K}$, for $\forall X \in \boldsymbol{S}_q$, and a vector of $P^q(X)$ which shows how many times a relative estimate of $\lambda_q(X), q \in \boldsymbol{K}$, is more than other relative estimates of $\lambda_k(X), k = \overline{1, K}$, we define a numerical expression of the priority of the qth criterion over other criteria of $k = \overline{1, K}$ as:

$$P^q(X) = \{p_k^q(X) = \frac{\lambda_q(X)}{\lambda_k(X)}, k = \overline{1, K}\}, p_k^q(X) \geq 1, \forall X \in \boldsymbol{S}_q \subset S, k = \overline{1, K}, \forall q \in \boldsymbol{K}. \tag{4.29}$$

Such a ratio $p_k^q(X) = \frac{\lambda_q(X)}{\lambda_k(X)}$. let us call the numerical expression of the priority of the q-th criterion over the rest of the criteria $k = \overline{1, K}$.

Definition 7a. ***On a given numerical expression of the priority of one criterion over***.

In the vector problem of Equations (3.1)–(3.4) with a priority of criterion of $q \in \boldsymbol{K}$ for $\forall X \in \boldsymbol{S}$, vector $P^q = \{p_k^q, k = \overline{1, K}\}$ is considered to be set by the person making decisions (i.e., decision-maker) if everyone is set a component of this vector. Set by the decision-maker, component p_k^q, from the point of view of the decision-maker, shows how many times a relative estimate of $\lambda_q(X), q \in \boldsymbol{K}$ is greater than other relative estimates of $\lambda_k(X), k = \overline{1, K}$. The vector of $p_k^q, k = \overline{1, K}$, is the numerical expression of the priority of the qth criterion over other criteria of $k = \overline{1, K}$:

$$P^q(X) = \{p_k^q(X), k = \overline{1, K}\}, p_k^q(X) \geq 1, \forall X \in \boldsymbol{S}_q \subset S, k = \overline{1, K}, \forall q \in \boldsymbol{K}. \tag{4.30}$$

The vector problem of Equations (3.1)–(3.4), in which the priority of any criteria is set, is called a vector problem with the set priority of criterion. The problem of a task of a vector of priorities arises when it is necessary to determine the point $X^o \in \boldsymbol{S}$ by the set vector of priorities.

In the comparison of relative estimates with a priority of criterion of $q \in \boldsymbol{K}$, as well as in a task with equivalent criteria, we define the additional numerical characteristic of λ which we call the *level*.

Definition 8. ***About the lower level among all relative estimates with a criterion priority***.

The λ level is the lowest among all relative estimates with a priority of criterion of $q \in \boldsymbol{K}$ such that:

$$\lambda \leq p_k^q \lambda_k(X),\ k = \overline{1, K},\ q \in \boldsymbol{K},\ \forall X \in \boldsymbol{S}_q \subset S; \tag{4.31}$$

The lower level for the performance of the condition in Equation (4.31) is defined as:

$$\lambda = \min_{k \in K} p_k^q \lambda_k(X), q \in \boldsymbol{K}, \forall X \in \boldsymbol{S}_q \subset S. \tag{4.32}$$

Equations (4.31) and (4.32) are interconnected and serve as a further transition from the operation of the definition of the minimum to restrictions, and vice versa. In Section 4, we gave the definition of a Pareto optimal point $X^o \in \boldsymbol{S}$ with equivalent criteria. Considering this definition as an initial one, we will construct a number of the axioms dividing an admissible set of $\boldsymbol{S}$ into, first, a subset of

Pareto optimal points $S°$, and, secondly, a subset of points $\boldsymbol{S}_q \subset \boldsymbol{S}, q \in \boldsymbol{K}$, with priority for the qth criterion.

4.2.2. Axiomatics of Criteria Priority in the Vector Optimization Problem.

Axiom 2. ***On a subset of points prioritized by criterion in a vector optimization problem***.

In the vector problems of mathematical programming of Equations (3.1)–(3.4), the subset of points $\boldsymbol{S}_q \subset \boldsymbol{S}$ is called the area of priority of criterion of $q \in \boldsymbol{K}$ over other criteria, if

$\forall X \in \boldsymbol{S}_q \ \forall k \in \boldsymbol{K} \ \lambda_q(X) \geq \lambda_k(X), q \neq k.$

This definition extends to a set of Pareto optimal points $\boldsymbol{S}^o$ that is given by the following definition.

Axiom 2a. ***About a subset of points, priority by criterion, on Pareto's great number in a vector problem***.

In a vector problem of mathematical programming the subset of points $\boldsymbol{S}_q^o \subset \boldsymbol{S}^o \subset$ S is called the area of a priority of criterion of $q \in \boldsymbol{K}$ over other criteria, if

$\forall \mathrm{X} \in S_q^o \ \forall k \in K \ \lambda_q(X) \geq \lambda_k(X), q \neq k.$

In the following we provide explanations.

Axiom 2 and 2a allow the breaking of the vector problems of mathematical programming in Equations (3.1)–(3.4) into an admissible set of points $\boldsymbol{S}$, including a subset of Pareto optimal points, $\boldsymbol{S}^o \subset \boldsymbol{S}$, and subsets:

One subset of points $\boldsymbol{S}' \subset \boldsymbol{S}$ where criteria are equivalent, and a subset of points of $\boldsymbol{S}'$ crossed with a subset of points $\boldsymbol{S}°$, allocated to a subset of Pareto optimal points at equivalent criteria $\boldsymbol{S}^{oo} = \boldsymbol{S}' \cap \boldsymbol{S}^o$. As will be shown further, this consists of one point of $X^o \in \boldsymbol{S}$, i.e.

$$X^o = \boldsymbol{S}^{oo} = \boldsymbol{S}' \cap \boldsymbol{S}^o, \boldsymbol{S}' \in \boldsymbol{S}, \boldsymbol{S}^o \subset \boldsymbol{S}; \quad (4.33)$$

"$\boldsymbol{K}$" subsets of points where each criterion of $q = \overline{1, K}$ has a priority over other criteria of $k = \overline{1, K}, q \neq k$, and thus breaks, first, sets of all admissible points $\boldsymbol{S}$, into subsets $\boldsymbol{S}_q \subset \boldsymbol{S}, q = \overline{1, K}$ and, second, a set of Pareto optimal points, $\boldsymbol{S}°$, into subsets $\boldsymbol{S}_q^o \subset \boldsymbol{S}_q \subset \boldsymbol{S}, q = \overline{1, K}$. This yields:

$$\boldsymbol{S}' \mathrm{U}\left(\cup_{q \in K} \boldsymbol{S}_q^o\right) \equiv \boldsymbol{S}°, \boldsymbol{S}_q^o \subset \boldsymbol{S}° \subset \boldsymbol{S}, q = \overline{1, K}. \quad (4.34)$$

We note that the subset of points $\boldsymbol{S}_q^o$, on the one hand, is included in the area (a subset of points) of priority of criterion of $q \in \boldsymbol{K}$ over other criteria: $\boldsymbol{S}_q^o \subset \boldsymbol{S}_q \subset \boldsymbol{S}$, and, on the other, in a subset of Pareto optimal points$\boldsymbol{S}_q^o \subset \boldsymbol{S}° \subset \boldsymbol{S}$.

Axiom 2 and the numerical expression of priority of criterion (Definition 5) allow the identification of each admissible point of $X \in \boldsymbol{S}$ (by means of vector:

$$P^q(X) = \{p_k^q(X) = \frac{\lambda_q(X)}{\lambda_k(X)}, k = \overline{1, K}\}, \text{ to form and choose:} \quad (4.35)$$

a subset of points by priority criterion $\boldsymbol{S}_q$, which is included in a set of points $\boldsymbol{S}$, $\forall q \in \boldsymbol{K}$ $X \in \boldsymbol{S}_q \subset \boldsymbol{S}$, (such a subset of points can be used in problems of clustering, but is beyond this article);

a subset of points by priority criterion $\boldsymbol{S}_q^o$, which is included in a set of Pareto optimal points $\boldsymbol{S}°$, $\forall q \in K, X \in \boldsymbol{S}_q^o \subset \boldsymbol{S}°$.

Thus, full identification of all points in the vector problem of Equations (3.1)–(3.4) is executed in sequence as:

Set of admissible points of $X \in \boldsymbol{S} \rightarrow$	Subset of points, optimum across Pareto, $X \in \boldsymbol{S}^o \subset \boldsymbol{S} \rightarrow$	Subset of points, optimum across Pareto $X \in \boldsymbol{S}_q^o \subset \boldsymbol{S}^o \subset \boldsymbol{S} \rightarrow$	Separate point of a $\forall X \in \boldsymbol{S}$ $X \in \boldsymbol{S}_q^o \subset \boldsymbol{S}^o \subset \boldsymbol{S}$

This is the most important result which allows the output of the principle of optimality and to construct methods of a choice of any point of Pareto's great number.

4.2.3. Principle of optimality 2. The solution of a vector problem with the set criterion priority in the VPMP

Definition 9. ***Principle of optimality 2. The solution of a vector problem with the set criterion priority in the VPMP***.

The vector problem of Equations (3.1)–(3.4) with the set priority of the qth criterion of $p_k^q \lambda_k(X), k = \overline{1, K}$ is considered solved if the point $X°$ and maximum level $\lambda°$ among all relative estimates is found such that:

$$\lambda^o = \boldsymbol{max}_{X\in S}\boldsymbol{min}_{k\in K} p_k^q \lambda_k(X), q \in \boldsymbol{K}. \quad (4.36)$$

Using the interrelation of Equations (4.31) and (4.32), we can transform the maximine problem of Equation (4.36) into an extreme problem of the form:

$$\lambda^o = \boldsymbol{max}_{X\in S}\, \lambda, \quad (4.37)$$

$$\text{at restriction } \lambda \le p_k^q \lambda_k(X), k = \overline{1,K}. \quad (4.38)$$

We call Equations (4.37) and (4.38) the λ-problem with a priority of the *q*th criterion.

The solution of the λ-problem is the point $\boldsymbol{X^o} = \{X^o, \lambda^o\}$. This is also the result of the solution of the vector problem of Equations (3.1)-(3.4) with the set priority of the criterion, solved on the basis of normalization of criteria and the principle of the guaranteed result.

In the optimum solution $\boldsymbol{X^o} = \{X^o, \lambda^o\}$, $X°$, an optimum point, and λ°, the maximum bottom level, the point of $X°$ and the λ° level correspond to restrictions of Equation (5.8), which can be written as: $\lambda^o \le p_k^q \lambda_k(X^o), k = \overline{1,K}$.

These restrictions are the basis of an assessment of the correctness of the results of a decision in practical vector problems of optimization.

From Definitions 1 and 2, "Principles of optimality", follows the opportunity to formulate the concept of the operation “opt”.

Definition 9. ***Mathematical operation "opt"*** **in the VPMP**.

In the vector problems of mathematical programming of Equations (3.1)–(3.4), in which "max" and "min" are part of the criteria, the mathematical operation "opt" consists of the definition of a point $X°$ and the maximum λ° bottom level to which all criteria measured in relative units are lifted:

$$\lambda° \le \lambda_k(X°) = \frac{f_k(X) - f_k^0}{f_k^* - f_k^0}, k = \overline{1,K}, \quad (4.39)$$

i.e., all criteria of $\lambda_k(X^o), k = \overline{1,K}$, are equal to or greater than the maximum level of λ° (therefore λ° is also called the guaranteed result).

Theorem 2. ***The theorem of the most inconsistent criteria in a vector problem with the set priority***. If in the convex vector problem of mathematical programming of Equations (3.1)–(3.4) the priority of the *q*th criterion of $p_k^q, k = \overline{1,K}, \forall q \in \boldsymbol{K}$ over other criteria is set, at a point of an optimum $X^o \in \boldsymbol{S}$ obtained on the basis of normalization of criteria and the principle of guaranteed result, there will always be two criteria with the indexes $r \in \boldsymbol{K}$, $t \in \boldsymbol{K}$, for which the following strict equality holds:

$$\lambda° = p_k^r \lambda_r(X°) = p_k^t\, \lambda_t(X°), r, t, \in \boldsymbol{K}, \quad (4.40)$$

and other criteria are defined by inequalities:

$$\lambda° \le p_k^q(X°), k = \overline{1,K}, \forall q \in \boldsymbol{K}, q \ne r \ne t\,. \quad (4.41)$$

Criteria with the indexes $r \in \boldsymbol{K}$, $t \in \boldsymbol{K}$ for which the equality of Equation (4.40) holds are called the most inconsistent.

Proof. Similar to Theorem 2 [19, 20].

We note that in Equations (4.40) and (4.41), the indexes of criteria $r, t \in \boldsymbol{K}$ can coincide with the $q \in \boldsymbol{K}$ index.

Consequence of Theorem 1, about equality of an optimum level and relative estimates in a vector problem with two criteria with a priority of one of them.

In a convex vector problem of mathematical programming with two equivalent criteria, solved on the basis of normalization of criteria and the principle of the guaranteed result, at an optimum point X^o equality is always carried out at a priority of the first criterion over the second:

$$\lambda^o = \lambda_1(X^o) = p_2^1(X°)\lambda_2(X^o), X^o \in \boldsymbol{S}\,, \quad (4.42)$$

where $p_2^1(X°) = \lambda_1(X^o)/\lambda_2(X^o)$, and at a priority of the second criterion over the first:

$$\lambda^o = \lambda_2(X^o) = p_1^2(X°)\lambda_1(X^o), X^o \in \boldsymbol{S},$$

where $p_1^2(X°) = \lambda_2(X^o)/\lambda_1(X^o)$.

4.2.4. Mathematical Method of the Solution of a Vector Problem with Criterion Priority.

Step 1. We solve a vector problem with equivalent criteria. The algorithm of the decision is presented in Section 4.1.4.

As a result of the decision, we obtain:

optimum points by each criterion separately $X_k^*, k = \overline{1,K}$ and sizes of criterion functions in these points of $f_k^* = f_k(X_k^*), k = \overline{1,K}$, which represent the boundary of a set of Pareto optimal points;

anti-optimum points by each criterion of $X_k^0 = \{x_j, j = \overline{1,N}\}$ and the worst unchangeable part of each criterion of $f_k^0 = f_k(X_k^0), k = \overline{1,K}$;

$\boldsymbol{X^o} = \{X^o, \lambda^o\}$, an optimum point, as a result of the solution of VPMP at equivalent criteria, i.e., the result of the solution of a maximine problem and the λ-problem constructed on its basis;

λ°, the maximum relative assessment which is the maximum lower level for all relative estimates of $\lambda_k(X^\circ)$, or the guaranteed result in relative units, λ° guarantees that all relative estimates of $\lambda_k(X^\circ)$ are equal to or greater than λ°:

$$\lambda^o \leq \lambda_k(X^o), k = \overline{1,K}, X^\circ \in S\ . \qquad (4.43)$$

The person making the decision carries out the analysis of the results of the solution of the vector problem with equivalent criteria.

If the received results satisfy the decision maker, then the process concludes, otherwise subsequent calculations are performed.

In addition, we calculate:

in each point $X_k^*, k = \overline{1,K}$ we determine sizes of all criteria of: $q = \overline{1,K}$

$\{f_q(X_k^*), q = \overline{1,K}\}, k = \overline{1,K}$, and relative estimates

$\lambda(X^*) = \{\lambda_q(X_k^*), q = \overline{1,K}, k = \overline{1,K}\},\ \lambda_k(X) = \frac{f_k(X)-f_k^0}{f_k^*-f_k^0}, \forall k \in \boldsymbol{K}$:

$$F(X^*) = \begin{vmatrix} f_1(X_1^*) \dots f_K(X_1^*) \\ \dots \\ f_1(X_K^*) \dots f_K(X_K^*) \end{vmatrix},\ \lambda(X^*) = \begin{vmatrix} \lambda_1(X_1^*) \dots \lambda_K(X_1^*) \\ \dots \\ \lambda_1(X_K^*) \dots \lambda_K(X_K^*) \end{vmatrix}. \qquad (4.44)$$

Matrices of criteria of $F(X^*)$ and relative estimates of $\lambda(X^*)$ show the sizes of each criterion of $k = \overline{1,K}$ upon transition from one optimum point $X_k^*, k\in\boldsymbol{K}$ to another $X_q^*, q\in\boldsymbol{K}$, i.e., on the border of a great number of Pareto.

at an optimum point at equivalent criteria X^o we calculate sizes of criteria and relative estimates:

$$f_k(X^o), k = \overline{1,K};\ \lambda_k(X^o), k = \overline{1,K}, \qquad (4.45)$$

which satisfy the inequality of Equation (4.43). In other points $X\in\boldsymbol{S^o}$, in relative units the criteria of $\lambda = \boldsymbol{min_{k\in K}}\lambda_k(X)$ are always less than λ°, given the λ-problem of Equations (4.37)-(4.38). This information is also a basis for further study of the structure of a great number of Pareto.

Step 2. Choice of priority criterion of $q \in \boldsymbol{K}$.

From theory (see Theorem 1) it is known that at an optimum point X^o there are always two most inconsistent criteria, $q \in \boldsymbol{K}$ and $v \in \boldsymbol{K}$, for which in relative units an exact equality holds:

$\lambda^o = \lambda_q(X^o) = \lambda_v(X^o), q, v \in \boldsymbol{K}, X\in \boldsymbol{S}$. Others are subject to inequalities:

$\lambda^o \leq \lambda_k(X^o), \forall k \in \boldsymbol{K},\ q \neq v \neq k$.

As a rule, the criterion which the decision-maker would like to improve is part of this couple, and such a criterion is called a priority criterion, which we designate $q \in \boldsymbol{K}$.

Step 3. Numerical limits of the change of the size of a priority of criterion $q \in \boldsymbol{K}$ are defined.

For priority criterion $q \in \boldsymbol{K}$ from the matrix of Equation (4.44) we define the numerical limits of the change of the size of criterion:

$$\text{in physical units of } f_k(X^o) \leq f_q(X) \leq f_q(X_q^*), k\in\boldsymbol{K}, \qquad (4.46)$$

where $f_q(X_q^*)$ derives from the matrix of Equation $F(X^*)$ (4.44), all criteria showing sizes measured in physical units, $f_k(X^o), k = \overline{1,K}$ from Equation (4.45), and,

$$\text{in relative units of } \lambda_k(X^o) \leq \lambda_q(X) \leq \lambda_q(X_q^*), k\in\boldsymbol{K}, \qquad (4.47)$$

where $\lambda_q(X_q^*)$derives from the matrix $\lambda(X^*)$, all criteria showing sizes measured in relative units (we note that $\lambda_q(X_q^*) = 1), \lambda_q(X^o)$ from Equation (4.44).

As a rule, Equations (4.46) and (4.47) are given for the display of the analysis.

Step 4. Choice of the size of priority criterion (decision-making).

The person making the decision carries out the analysis of the results of calculations of Equation 5.14) and from the inequality of Equation (4.46) chooses the numerical size f_q of the criterion of $q \in \boldsymbol{K}$:

$$f_q(X^o) \leq f_q \leq f_q(X^*_q), q \in \boldsymbol{K}. \quad (4.48)$$

For the chosen size of the criterion of f_q it is necessary to define a vector of unknown X^{oo}. For this purpose, we carry out the subsequent calculations.

Step 5. Calculation of a relative assessment.

For the chosen size of the priority criterion of f_q the relative assessment is calculated as:

$$\lambda_q = \frac{f_q - f_q^0}{f_q^* - f_q^0}, \quad (4.49)$$

which upon transition from point X^o to X^*_q, according to Equation (4.44), lies in the limits:

$$\lambda_q(X^\circ) \leq \lambda_q \leq \lambda_q(X^*_q) = 1.$$

Step 6. Calculation of the coefficient of linear approximation.

Assuming a linear nature of the change of criterion of $f_q(X)$ in Equation (4.49) and according to the relative assessment of $\lambda_q(X)$, using standard methods of linear approximation we calculate the proportionality coefficient between$\lambda_q(X^\circ), \lambda_q$, which we call ρ:

$$\rho = \frac{\lambda_q - \lambda_q(X^o)}{\lambda_q^* - \lambda_q^0}, q \in \boldsymbol{K}.$$

Step 7. Calculation of coordinates of priority criterion with the size f_q.

In accordance with Equation (4.48.4), the coordinates of the X_q priority criterion point lie within the following limits: $X^o \leq X_q \leq X^*_q, q \in \boldsymbol{K}$. Assuming a linear nature of change of the vector

$X_q = \{x_1^q, \dots, x_N^q\}$ we determine coordinates of a point of priority criterion with the size f_q with the relative assessment of Equation (4.45):

$$X_q = \{x_1^q = x_1^o + \rho(x_q^*(1) - x_1^o), \dots, \\ x_N^q = x_N^o + \rho(x_q^*(N) - x_N^o)\}, \quad (4.50)$$

where $X^o = \{x_1^o, \dots, x_N^o\}, X_q^* = \{x_q^*(1), \dots, x_q^*(N)\}$.

Step 8. Calculation of the main indicators of a point x_q (5.20).

For the obtained point x_q, we calculate:

all criteria in physical units: $F^q = \{f_k(x^q), k = \overline{1, K}\}$;

all relative estimates of criteria:

$$\lambda^q = \{\lambda_k^q, k = \overline{1, K}\}, \lambda_k(x^q) = \frac{f_k(x^q) - f_k^0}{f_k^* - f_k^0}, k = \overline{1, K}; \quad (4.51)$$

the vector of priorities: $P^q = \{p_k^q = \frac{\lambda_q(x^q)}{\lambda_k(x^q)}, k = \overline{1, K}\}$;

the maximum relative assessment: $\lambda^{oq} = \min\,(p_k^q \lambda_k(x^q), k = \overline{1, K})$.

Any (5.21) point from Pareto's set $\boldsymbol{X_t^o} = \{\lambda_t^o, X_t^o\} \in \boldsymbol{S^o}$ can be similarly calculated.

Analysis of results. The calculated size of criterion $f_q(X_t^o), q \in \boldsymbol{K}$ is usually not equal to the set f_q. The error of the choice of $\Delta f_q = |f_q(X_t^o) - f_q|$ is defined by the error of linear approximation. The results of the study of *symmetry* in VPMP with a given priority are similar as for VPMP with equivalent criteria, but the center of symmetry is shifted towards the priority criterion.

Conclusion on the theory and axiomatics of vector optimization.

The presented theory, axiomatics, principles of optimality are a further development of the axiomatic approach laid down in the famous work "Elements" by the ancient Greek scientist Euclid, who presented axioms for one-dimensional mathematics. This is reflected in the theory of optimization with one criterion. Axiomatics (Mashunin Yu.K.), set forth in the work, is aimed at a systematic (with many criteria) study of objects, processes of engineering systems.

5. Software and Methodology for Modeling and Making an Optimal Decision on the Selection of Parameters of Complex Engineering Systems

5.1. Software for modeling complex engineering systems based on the theory and methods of vector optimization

Mathematical models of the structure of the material (2.4) - (2.8), (2.22)-(2.27) and the engineering systems are built in the form of a vector problem of nonlinear programming (VPNP). We will present software for modeling engineering systems based on the theory and methods of solving vector optimization problems, [46].

5.1.1. Software development of the VPNP solution

The software for solving the vector problem of nonlinear programming (3.1) -(3.4), on the basis of which models of engineering systems are formed, is implemented on the basis of the algorithm for solving the VPNP, described in the previous sections. In solving the VPNP for each criterion, the FMINCON program (...) in the MATLAB system was used.

When using the FMINCON(...) program, it is necessary to develop two subroutines – functions. The first function includes two blocks: the first block is designed to evaluate at point X the criterion $f_k(X)$ $\forall k \in$К; the second block to calculate the first derivative at this point $\frac{df_k(X)}{dX}$ $\forall k \in$К. The second function includes the same two blocks for constraints only. The FMINCON program (...) is used in the first step of the algorithm (maximizing the criteria) and in the second step of the algorithm (minimization). Similarly, according to the algorithm, step 4 and 5 are solved λ-problem.

In general, with nonlinear constraints, the software for solving the VPNP includes

K*2 (1 step) + K*2 (2 step) + 2 (λ-problem) functions. Since the criteria and limitations of the VPNP are individual, individual software is written for each VPNP. To solve the VPNP (3.1) - (3.4) below is a program that essentially represents a program - a template for writing and solving other vector problems of nonlinear programming (3.1) - (3.4) - mathematical models of engineering systems.

5.1.2. Numerical implementation of a vector problem of nonlinear programming

Example 4.1.

It is given. The consideration of the vector nonlinear (convex) programming problem with four homogeneous criteria. In terms of criteria, we use a circle, and with the linear restrictions the problem is therefore solved orally and imposed on variables.

$$opt\ F(X) = \{min\ F_2(X) = \{min\ f_1(X) \equiv (x_1 - 80)^2 + (x_2 - 80)^2, \quad (5.1)$$

$$min\ f_2(X) \equiv (x_1 - 80)^2 + (x_2 - 20)^2, \quad (5.2)$$

$$min\ f_3(X) \equiv (x_1 - 20)^2 + (x_2 - 20)^2, \quad (5.3)$$

$$min\ f_4(X) \equiv (x_1 - 20)^2 + (x_2 - 80)^2\}, \quad (5.4)$$

$$\text{at restrictions } 0 \le x_1 \le 100,\ 0 \le x_2 \le 100. \quad (5.5)$$

Need to be determined. Develop software in MATLAB solutions vector problem nonlinear programming. Using software solve the problem (5.1)-(5.5).

5.1.3. Software for solving the vector problem of nonlinear programming (VPNP)

To solve the vector problem of nonlinear programming (5.1) - (5.5) - a model of an engineering system, a program has been developed in the MATLAB system, which implements an algorithm for solving VPNP with equivalent criteria. The following is the result of the VPNP (5.1) to (5.5) decision obtained by the program.

Recording the program in MATLAB format

```
% Программа "Решение векторной задачи нелинейного программирования":
 function [x,f] = VPNP_2_4Krit_100(x)
% Автор: Машунин Юрий Константинович (Mashunun Yu. K.)
% Алгоритм и программа предназначена для использования в образовании и научных
% исследованиях, для коммерческого использования обращаться: Mashunin@mail.ru
% Algorithm VPNP: 4Kritery + L-zadaha
%[X,Fval,EXITFLAG,OUTPUT,LAMBDA,GRAD,HESSIAN]=
%                                    FMINCON(FUN,Xo,A,b,Aeq,beq, lb,ub,nonlcon,op-
tions,P1,P2,...)
disp('*** Блок Исходных данных. ВЗНП:***')
disp('opt F(X)={max F1(X)={min f1=(x1-80).^2+(x2-80).^2; ')
disp('                     min f2=(x1-80).^2+(x2-20).^2; ')
disp('                     min f3=(x1-20).^2+(x2-20).^2; ')
disp('                     min f4=(x1-20).^2+(x2-80).^2; ')
disp('                              0<=x1<=100, 0<=x2<=100 ')
lb=[0. 0.];
ub=[100. 100.]; Xo=[0. 0.];
options=optimset('LargeScale','off');
options=optimset(options,'GradObj','on','GradConst','off');
A=[1 0;
```

```
   0 1];
b=[100 100];
Aeq=[]; beq=[];
XoK1max=[0. 0.];
disp('*** Шаг 1. Решение по каждому критерию (наилучшее) ***')%
[x1max,f1max]= fmincon('VPNP_2_Krit1max',XoK1max,A,b,Aeq,beq,lb,ub,'',options)
[f1X1max] = VPNP_2_Krit1min(x1max)
[f2X1max] = VPNP_2_Krit2min(x1max)
[f3X1max] = VPNP_2_Krit3min(x1max)
[f4X1max] = VPNP_2_Krit4min(x1max)
XoK2max=[0. 0.];
[x2max,f2max]= fmincon('VPNP_2_Krit2max',XoK2max,A,b,Aeq,beq,lb,ub,'',options)
[f1X2max] = VPNP_2_Krit1min(x2max)
[f2X2max] = VPNP_2_Krit2min(x2max)
[f3X2max] = VPNP_2_Krit3min(x2max)
[f4X2max] = VPNP_2_Krit4min(x2max)
XoK3max=[0. 0.];
[x3max,f3max]= fmincon('VPNP_2_Krit3max',XoK3max,A,b,Aeq,beq,lb,ub,'',options)
[f1X3max] = VPNP_2_Krit1min(x3max)
[f2X3max] = VPNP_2_Krit2min(x3max)
[f3X3max] = VPNP_2_Krit3min(x3max)
[f4X3max] = VPNP_2_Krit4min(x3max)
XoK4max=[0. 0.];
[x4max,f4max]= fmincon('VPNP_2_Krit4max',XoK4max,A,b,Aeq,beq,lb,ub,'',options)
[f1X4max] = VPNP_2_Krit1min(x4max)
[f2X4max] = VPNP_2_Krit2min(x4max)
[f3X4max] = VPNP_2_Krit3min(x4max)
[f4X4max] = VPNP_2_Krit4min(x4max)
disp('*** Шаг 2. Решение по каждому критерию (наихудшее) ***')%
XoK1min=[0. 0.];
[x1min,f1min]= fmincon('VPNP_2_Krit1min',XoK1min,A,b,Aeq,beq,lb,ub,'',options)
[f1X1min] = VPNP_2_Krit1min(x1min)
[f2X1min] = VPNP_2_Krit2min(x1min)
[f3X1min] = VPNP_2_Krit3min(x1min)
[f4X1min] = VPNP_2_Krit4min(x1min)
[x2min,f2min] = fmincon('VPNP_2_Krit2min',Xo,A,b,Aeq,beq,lb,ub,'',options)
[f1X2min] = VPNP_2_Krit1min(x2min)
[f2X2min] = VPNP_2_Krit2min(x2min)
[f3X2min] = VPNP_2_Krit3min(x2min)
[f4X2min] = VPNP_2_Krit4min(x2min)
[x3min,f3min] = fmincon('VPNP_2_Krit3min',Xo,A,b,Aeq,beq,lb,ub,'',options)
[f1X3min] = VPNP_2_Krit1min(x3min)
[f2X3min] = VPNP_2_Krit2min(x3min)
[f3X3min] = VPNP_2_Krit3min(x3min)
[f4X3min] = VPNP_2_Krit4min(x3min)
[x4min,f4min] = fmincon('VPNP_2_Krit4min',Xo,A,b,Aeq,beq,lb,ub,'',options)
[f1X4min] = VPNP_2_Krit1min(x4min)
[f2X4min] = VPNP_2_Krit2min(x4min)
[f3X4min] = VPNP_2_Krit3min(x4min)
[f4X4min] = VPNP_2_Krit4min(x4min)
disp('*** Шаг 3. Системный анализ результатов ***')%
disp('Оценка критериев в точках оптимума: X1min,X2min,X3min,X4min')%
F=[f1X1min f2X1min f3X1min f4X1min;
   f1X2min f2X2min f3X2min f4X2min;
   f1X3min f2X3min f3X3min f4X3min;
   f1X4min f2X4min f3X4min f4X4min]
d1=f1X1min-f1X1max
d2=f2X2min-f2X2max
d3=f3X3min-f3X3max
d4=f4X4min-f4X4max
disp('Оценка критериев в относительных единицах: X1min,X2min,X3min,X4min')%
L=[(f1X1min-f1X1max)/d1   (f2X1min-f2X2max)/d2   (f3X1min-f3X3max)/d3   (f4X1min-
f4X4max)/d4;
   (f1X2min-f1X1max)/d1   (f2X2min-f2X2max)/d2   (f3X2min-f3X3max)/d3   (f4X2min-
f4X4max)/d4;
```

```
   (f1X3min-f1X1max)/d1   (f2X3min-f2X2max)/d2   (f3X3min-f3X3max)/d3   (f4X3min-
f4X4max)/d4;
   (f1X4min-f1X1max)/d1   (f2X4min-f2X2max)/d2   (f3X4min-f3X3max)/d3   (f4X4min-
f4X4max)/d4]
disp('*** Шаг 4. Построение L-задачи ***')%
Ao=[1 0 0;
    0 1 0];
bo=[100 100]; Aeq=[]; beq=[];
Xoo=[0 0 0]
lbo=[0. 0. 0.]
ubo=[100. 100. 1]
disp('*** Шаг 5. Решение L-задачи ***')%
[Xo,Lo]=fmincon('VPNP_2_L',Xoo,Ao,bo,Aeq,beq,lbo,ubo,'VPNP_2_LConst') %,options)
disp('Оценка критериев в точке оптимума Xo')%
[f1Xo] = VPNP_2_Krit1min(Xo(1:2))
[f2Xo] = VPNP_2_Krit2min(Xo(1:2))
[f3Xo] = VPNP_2_Krit3min(Xo(1:2))
[f4Xo] = VPNP_2_Krit4min(Xo(1:2))
disp('Оценка критериев в точке оптимума Xo в относительных единицах')%
L1Xo=(f1Xo+f1max)/d1
L2Xo=(f2Xo+f2max)/d2
L3Xo=(f3Xo+f3max)/d3
L4Xo=(f4Xo+f4max)/d4
% ****Конец*********
%    [Программа "Расчет 1 критер. - max"] файл: VPNP_2_Krit1max
 function [f,G] = VPNP_2_Krit1max(x)
f=-(x(1)-80).^2-(x(2)-80).^2;     %Расчет функции - критерий 1
G=[-2*(x(1)-80),    -2*(x(2)-80)];%Расчет 1 производной критерия 1
%  [Программа "Расчет 1 критер. - min"] Файл: VPNP_2_Krit1min
 function [f,G] = VPNP_2_Krit1min(x);
f=(x(1)-80).^2+(x(2)-80).^2;
G=[2*(x(1)-80);   2*(x(2)-80)];
% [Программа "Расчет 2 критер. - max"] Файл:  VPNP_2_Krit2max
 function [f,G] = VPNP_2_Krit2max(x);
f=-(x(1)-80).^2-(x(2)-20).^2;
G=[-2*(x(1)-80);    -2*(x(2)-20)];
%    [Программа "Расчет критер. 2 - min"] Файл: VPNP_2_Krit2min
 function [f,G] =VPNP_2_Krit2min(x);
f=(x(1)-80).^2+(x(2)-20).^2;
G=[2*(x(1)-80);   2*(x(2)-20)];
% [Программа "Расчет 3 критер. - max"] Файл: VPNP_2_Krit3max
 function [f,G] = VPNP_2_Krit3max(x);
f=-(x(1)-20).^2-(x(2)-20).^2;
G=[-2*(x(1)-20);    -2*(x(2)-20)];
% [Программа "Расчет 3 критер. - min"] Файл: VPNP_2_Krit3min
 function [f,G] = VPNP_2_Krit3min(x);
f=(x(1)-20).^2+(x(2)-20).^2;
G=[2*(x(1)-20);    2*(x(2)-20)];
% [Программа "Расчет 4 критер. - max"] Файл:VPNP_2_Krit4max
function [f,G] = VPNP_2_Krit4max(x);
f=-(x(1)-20).^2-(x(2)-80).^2;
G=[-2*(x(1)-20);    -2*(x(2)-80)];
% [Программа "Расчет 4 критер. - max"] Файл:VPNP_2_Krit4max
function [f,G] = VPNP_2_Krit4max(x);
f=-(x(1)-20).^2-(x(2)-80).^2;
G=[-2*(x(1)-20);    -2*(x(2)-80)];
%[Программа "Расчет критер. L-задачи"] файл: VPNP_2_L
function [f,G] = VPNP_2_L(x)
f=-x(3);
G=[0;    0;  -1];
%        [Программа "Расчет ограничений L-задачи"] файл: VPNP_1_LConst
function [c,ceq,DC,DCeq]= VPNP_2_LConst(x)
d1=12800;d2=12800;d3=12800;d4=12800;
f1X1max=12800;f2X2max=12800;f3X3max=12800;f4X4max=12800;
c(1)=((x(1)-80).^2+(x(2)-80).^2)/d1+x(3)-f1X1max/d1;
c(2)=((x(1)-80).^2+(x(2)-20).^2)/d2+x(3)-f2X2max/d2;
```

```
c(3)=((x(1)-20).^2+(x(2)-20).^2)/d3+x(3)-f3X3max/d3;
c(4)=((x(1)-20).^2+(x(2)-80).^2)/d4+x(3)-f4X4max/d4;
G1=[2*(x(1)-80)/d1, 2*(x(1)-80)/d2, 2*(x(1)-20)/d3, 2*(x(1)-20)/d4;
    2*(x(2)-80)/d1, 2*(x(2)-20)/d2, 2*(x(2)-20)/d3, 2*(x(2)-80)/d4;
            1.0,            1.0,           1.0,          1.0];
ceq=[]; DCeq=[];
    % *****Конец*****
```

% *****Конец*****

5.1.4. Solving the vector problem of nonlinear programming (5.1) - (5.5)

The above program is used to solve the VPNP (5.1) to (5.5). The solution is presented as a sequence of steps.

Step 1. The vector problem (5.1) – (5.5) on max for each criterion is solved separately. Results of the decision of the VPMP (5.1) to (5.5) for each criterion:

Criterion 1: $X_1^* = \{x_1 = 0, x_2 = 0\}, f_1^* = f_1(X_1^*) = -12800;$ (5.6)

Criterion 2.$X_2^* = \{x_1 = 0, x_2 = 100\}, f_2^* = f_2(X_2^*) = -12800;$

Criterion 3: $X_3^* = \{x_1 = 100, x_2 = 100\}, f_3^* = f_3(X_3^*) = -12800:$

Criterion 4: $X_4^* = \{x_1 = 100, x_2 = 0\}, f_4^* = f_4(X_4^*) = -12800.$

Step 2. The vector problem (5.1) - (5.5) on min for each criterion is solved separately. Results of the decision of the VPMP (5.1) to (5.5) for each criterion:

Criterion 1: $X_1^0 = \{x_1 = 80, x_2 = 80\}, f_1^0 = f_1(X_1^0) = 0;$ (5.7)

Criterion 2. $X_2^0 = \{x_1 = 80, x_2 = 20\}, f_2^0 = f_2(X_2^0) = 0;$

Criterion 3: $X_3^0 = \{x_1 = 20, x_2 = 20\}, f_3^0 = f_3(X_3^0) = 0:$

Criterion 4: $X_4^0 = \{x_1 = 20, x_2 = 80\}, f_4^0 = f_4(X_4^0) = 0.$

The results of the decision VPMP (5.1) to (5.5) in each criterion in the field of restrictions (5.6) are presented in Fig. 5.1 at salient points.

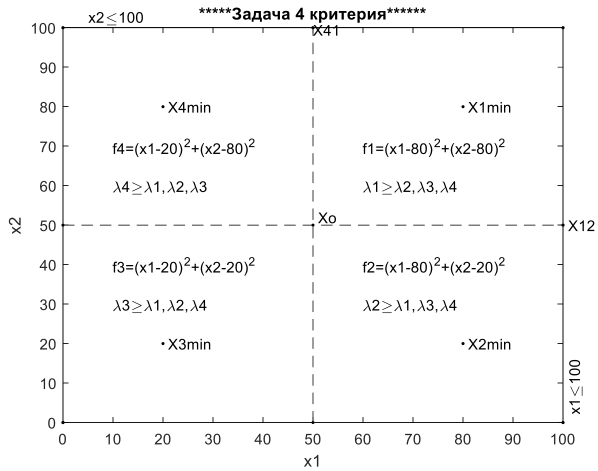

Figure 5.1. Limitations of VPNP (5.1) - (5.5), optimum points $X_k^* = \{X1min, X2min, X3min, X4min\}$ and relative estimates $\lambda_k(X), k = \overline{1,K}, K = 4$

The Pareto set lies between optimum points $X_1^* X_2^* X_3^* X_4^*$, i.e., the area of admissible points of $\boldsymbol{S}$ formed by restrictions (5.5) coincides with a point set, which is Pareto-optimal $\boldsymbol{S}^o$, $\boldsymbol{S}^o = \boldsymbol{S}$.

Step 3. A system analysis of the set of Pareto points is performed. At the optimum points $X^* = \{X_k^*, k = \overline{1,K}\}$, the values of the target functions $F(X^*)$ and the relative values $\lambda(X^*)$ are determined:

$F(X^*) = \{\{f_q(X_k^*), q = \overline{1,K}\}, k = \overline{1,K}\},$

$\lambda(X^*) = \{\{\lambda_q(X_k^*), q = \overline{1,K}\}, k = \overline{1,K}\},$

In the MATLAB system, at the optimum points: $X_k^* = \{X1min, X2min, X3min, X4min\}$, the calculation of these functions will be as follows (System Analysis Result):

$$F(X^*) = \begin{vmatrix} f_1(X_1^*)\ f_2(X_1^*)\ f_3(X_1^*)\ f_4(X_1^*) \\ f_1(X_2^*)\ f_2(X_2^*)\ f_3(X_2^*)\ f_4(X_2^*) \\ f_1(X_3^*)\ f_2(X_3^*)\ f_3(X_3^*)\ f_4(X_3^*) \\ f_1(X_4^*)\ f_2(X_4^*)\ f_3(X_4^*)\ f_4(X_4^*) \end{vmatrix} = \begin{vmatrix} 0\ 3600\ 7200\ 3600 \\ 3600\ 0\ 3600\ 7200 \\ 7200\ 3600\ 0\ 3600 \\ 3600\ 7200\ 3600\ 0 \end{vmatrix},$$

$$\lambda(X^*) = \begin{vmatrix} \lambda_1(X_1^*) & \lambda_2(X_1^*) & \lambda_3(X_1^*) & \lambda_4(X_1^*) \\ \lambda_1(X_2^*) & \lambda_2(X_2^*) & \lambda_3(X_2^*) & \lambda_4(X_2^*) \\ \lambda_1(X_3^*) & \lambda_2(X_3^*) & \lambda_3(X_3^*) & \lambda_4(X_4^*) \\ \lambda_1(X_4^*) & \lambda_2(X_4^*) & \lambda_3(X_4^*) & \lambda_4(X_4^*) \end{vmatrix} = \begin{vmatrix} 1.0 & 0.7188 & 0.4375 & 0.7188 \\ 0.7188 & 1.0 & 0.7188 & 0.4375 \\ 0.4375 & 0.7188 & 1.0 & 0.7188 \\ 0.7188 & 0.4375 & 0.7188 & 1.0 \end{vmatrix}.$$

At points of an optimum of $X_k^*, k=\overline{1,K}$all relative estimates (the normalized criteria) are equal to the unit:

$\lambda_k(X_k^*) = \frac{f_k(X_k^*)-f_k^0}{f_k^*-f_k^0} = 1, k = \overline{1,K}, K = 4.$

At points of an optimum of $X_k^0, k=\overline{1,K}$ (anti optimum), all relative estimates are equal to zero:

$\lambda_k(X_k^0) = \frac{f_k(X_k^0)-f_k^0}{f_k^*-f_k^0} = 0, k = \overline{1,K}, K = 4.$.

From here, $\forall k \in \boldsymbol{K}, \forall X \in \boldsymbol{S}, 0 \le \lambda_k(X) \le 1$.

Step 4. Builds λ-problem.

Step 5. Solution λ-problem. Results of the solution λ-problems:

Xo=$X^o = \{x_1 = 50.0, x_2 = 50.0, x_3 = 0.8594\}$ – the optimum point where x_3=λ^o;

x_1, x_2 corresponds to the x_1, x_2 problem (5.1) to (5.5);

Lo = λ^o= 0.8594 represents the optimal value of the objective function. The functions $\lambda_1(X), \lambda_2(X), \lambda_3(X), \lambda_4(X)$, as well as the optimum points X^o and λ^o, which are obtained at their intersection, in the three-dimensional coordinate system x_1, x_2, λ are shown in Figure 5.2.

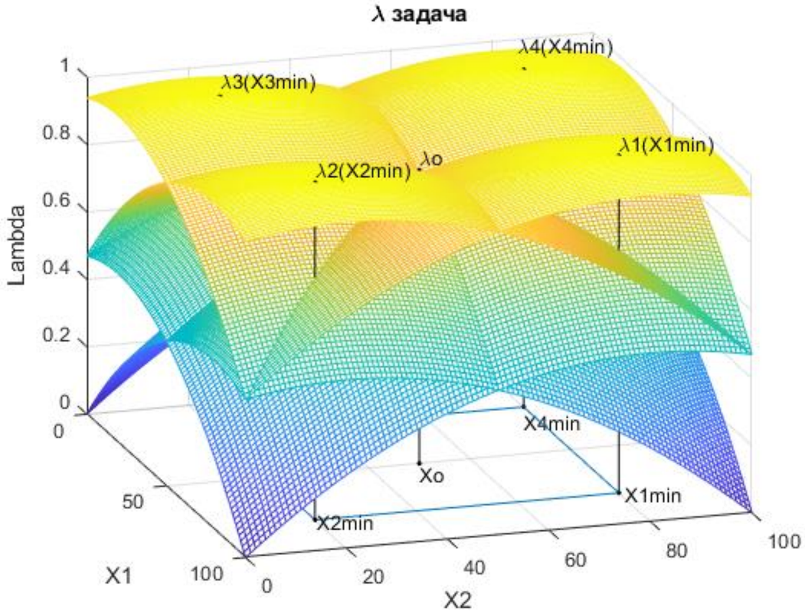

Figure 5.2. The results of the solution to VPMP: $\lambda_1(X), \lambda_2(X), \lambda_3(X), \lambda_4(X)$;
The optimum points X^o; the relative estimates λ^o

In Fig. 5.1, 5.2 shows that the region (set of points) bounded by points:

$\boldsymbol{S_q} = \{X_1^* = X_{1opt} X_{12} X^o X_{41}\}$ is characterized by the fact that $\lambda_1(X) \ge \lambda_k(X), k = \overline{2,4}, X \in \boldsymbol{S_1}$, (Figure 4.1 shows how λ1>λ2, λ3, λ4, i.e. the area of $\boldsymbol{S_{q=1}}$ is preceded by the first criterion. In this area, the priority of the first criterion over the others is always greater than one:

$p_k^1(X) = \lambda_1(X)/\lambda_k(X) \ge 1, \forall X \in \boldsymbol{S_1}$.

Similarly, the areas (sets of points) given priority by the corresponding criterion are shown, together they give a set of points optimal according to Pareto, $\boldsymbol{S^o}$, and it (for this example) is equal to the set of allowable points: $\boldsymbol{S^o} = \boldsymbol{S_1^o} \cup \boldsymbol{S_2^o} \cup \boldsymbol{S_3^o} \cup \boldsymbol{S_4^o} \cup X^o = \boldsymbol{S}$.

If we solve the problem (5.1) - (5.5) with two criteria, for example, third and fourth, the set of Pareto-optimal points lies on the segment $X_3^* X_4^*$, and the X^{oo} point determines the result of the solution. λ^{oo} is the maximum level, and $\lambda^{oo} = \lambda_3(X^{oo}) = \lambda_4(X^{oo}) = 0.7917$ according to theorem 1.

The Pareto set lies between the points of the optimum$X_1^* X_2^* X_3^* X_4^*$, i.e. the region of allowable points S formed by constraints (5.5) coincides with the set of Pareto optimal points $\boldsymbol{S^o}$, $\boldsymbol{S^o} = \boldsymbol{S}$.

5.2. Methodology for Modeling and Making an Optimal Decision on the Choice of Parameters of Complex Engineering Systems in Conditions of Certainty, Uncertainty

As an object of research, we consider "Engineering systems", which include "technical systems", "technological processes", "materials", [17, 42, 44]. The study of the engineering system is carried out, firstly, under conditions of certainty, when the data on the functional characteristics of the engineering system are known; secondly, under conditions of uncertainty, when discrete values of individual characteristics are known; There is also data on the restrictions that are imposed on the functioning of the system. Mathematical software is based on the methods of vector optimization presented in the third section. Methodological support for modeling the engineering system is formed as: "Methodology for Modeling and Making an Optimal Decision on the Choice of Parameters of Complex Engineering Systems in Conditions of Certainty, Uncertainty".

5.2.1. Types of Problems Arising in the Process of Modeling and Making an Optimal Decision on the Selection of Parameters of Complex Engineering Systems

The problems that arise in the process of making an optimal decision on the selection of optimal parameters of complex Engineering systems on the basis of vector optimization include three types sequentially.

1 type. *Solution of a vector problem of mathematical programming with equivalent criteria*. The result obtained is the basis for further research of the system. In this case, the method of solving a vector problem with equivalent criteria is used. If the result obtained satisfies the decision-maker (decision-maker - designer), then it is taken as a basis. If it does not satisfy, then move on to the second type (direct problem) or the third type of solving vector problems (Inverse problem).

2 type. *The solution of the direct problem of vector optimization*, which consists in the following: "What will be the indicators (characteristics) if the parameters of complex engineering systems are changed." - The method of solving a vector problem with equivalent criteria is used.

3 *type*. The solution of the inverse problem of vector optimization, which consists in the following: "What will be the parameters of complex engineering systems with given characteristics". - A method for solving a vector problem at a given criterion priority is used.

5.2.2. Methodology for Modeling and Making an Optimal Decision on the Choice of Parameters of Complex Engineering Systems in Conditions of Certainty, Uncertainty

The methodology includes three blocks, divided into a number of stages.

Block 1. ***The formation of technical specifications***, the transformation of uncertainty conditions (related to experimental data) into conditions of certainty, the construction of a mathematical and numerical model of an engineering system (the process of modeling of an engineering system) includes 4 stages.

Stage 1. Formation of technical specifications (initial data) for numerical modeling and selection of optimal system parameters. The initial data is formed by the designer who designs the engineering system.

Stage 2. Construction of mathematical and numerical model of the engineering system in conditions of certainty and uncertainty.

Stage 3. Transformation of uncertainty conditions into certainty conditions and construction of a mathematical and numerical model of an engineering system under conditions of certainty.

Stage 4. Construction of an aggregated mathematical and numerical model of an engineering system under conditions of certainty

Block 2. Methodology of the process of optimal decision making (selection of optimal parameters) in an engineering system based on vector optimization (the process of simulation of an engineering system)

Stage 5. Solution of a vector problem of mathematical programming (VPMP) - a model of an engineering system with equivalent criteria (solution of a direct problem).

Stage 6. Geometric interpretation of the results of the vector problem of mathematical programming solution with N parameters and K criteria into a two-dimensional coordinate system in relative units.

Stage 7. Solution of a vector problem of mathematical programming - a model of an engineering system with a given priority of the criterion (solution of the inverse problem).

Block 3. Research, design, geometric interpretation and selection of optimal parameters of a complex engineering system in multidimensional mathematics includes 2 stages.

Stage 8. Geometric interpretation of the solution results in the design of an engineering system for the transition from two-dimensional to *N*-dimensional space in relative units.

Stage 9. Geometric interpretation of the solution results in the design of an engineering system for the transition from two-dimensional to *N*-dimensional space in physical units.

6. Selection of optimal parameters of a material of a complex structure under conditions of certainty and uncertainty on the basis of multidimensional mathematics. Numerical implementation.

The numerical realization of the choice of optimal parameters of a material of a complex structure is carried out in accordance with the theoretical foundations of multidimensional mathematics, including axiomatics, principles of optimality and constructive methods of multidimensional mathematics both with equivalent criteria and with a given priority of criteria, in accordance with Section 2.

The methodology of the process of making an optimal decision (selection of optimal parameters) of an engineering system, including the material of a complex structure, is set out in Section 5. The problem of decision-making in a complex structure of the material is considered, which is known to:

first, data on the functional relationship of several characteristics with its components (conditions of certainty); secondly, data on a certain set of discrete values of several characteristics (experimental results), in relation to discrete values of parameters – experimental data (uncertainty conditions); thirdly, the restrictions imposed on the functioning of the material of a complex structure.

The numerical problem of modeling a material of a complex structure is considered with equivalent criteria and with a given priority of the criterion.

6.1. Block 1. Formation of technical specifications and construction of a mathematical and numerical model of a material of complex structure (the process of modeling of the structure (composition) of the material).

The first stage, as well as the stage of analyzing the results of the solution, choosing the priority criterion and its value, is performed by the constructor of a material of a complex structure. The remaining stages are performed by a mathematician-programmer.

6.1.1. Stage 1. ***The technical assignment: "The choice of optimum parameters of material"***

It is given. Material which structure is defined by four components: $Y = \{y_1, y_2, y_3, y_4\}$– a vector (operated) variable. *Y* values represent a vector of managed variables. The parameters of the material structure are defined, which vary within the following limits:

$21 \le y_1 \le 79;\ 5 \le y_2 \le 59;\ 2.1 \le y_3 \le 9.0;\ 2.2 \le y_4 \le 7.0.$

Input data for a decision-making are four characteristics:

The functioning of the structure of the material is determined by four characteristics (criteria): $H(Y) = \{h_1(Y), h_2(Y), h_3(Y), h_4(Y)\}$, the value of which depends on the parameters : $Y = \{y_1, y_2, y_3, y_4\}$.

Conditions of certainty. For the first characteristic: $h_1(Y)$ the functional dependence on the parameters $Y = \{y_v, v = \overline{1, V}, V = 4\}$ is known:

$$h_1(Y) = 323.84 - 2.249y_1 - 3.49y_2 + 10.7267y_3 + 13.124y_4 + 0.0968y_1y_2 - 0.062y_1y_3 - 0.169y_1y_4 + 0.0743y_2y_3 - 0.1042y_2y_4 - 0.0036y_3y_4 + 0.0143y_1^2 + 0.0118y_2^2 - 0.2434y_3^2 - 0.5026x_4^2. \quad (6.1)$$

Functional limitations imposed on the third characteristic (property) of the material:

$$\min h_3(X) = 92.4,\ \max h_3(X) = 161.5. \quad (6.2).$$

Conditions of uncertainty. ***The results of the experimental data*** are known: for the second, third and fourth characteristics $h_k(Y), k = 2, 3, 4$ for the corresponding values of the parameters: $Y = \{y_v = \{y_{vi},\ i = \overline{1, M}\}, v = \overline{1, V}\}$.

The numerical values of the parameters *Y* and the characteristics $h_2(Y), h_3(Y), h_4(Y)$ are presented in Table 2.

Table 2. Experimental values of the parameters y_1, y_2, y_3, y_4 and characteristics of the structure of the material $h_2(Y), h_3(Y), h_4(Y)$.

y_1	y_2	y_3	y_4	$h_1(Y)$	$h_2(Y)$	$h_4(Y)$
20	0	2	2	1149.6	115.1	24.24
20	0	2	5	1164.0	114.5	27.60
20	0	2	8	1176.0	114.4	28.80
20	0	5	2	1212.0	118.8	30.00
20	0	5	5	1260.0	113.8	31.20
20	0	5	8	1257.6	113.3	32.40
20	0	8	2	1256.4	110.7	33.60
20	0	8	5	1252.8	109.2	34.80
20	0	8	8	1251.6	108.5	34.80
20	30	2	2	2143.2	128.3	19.92
20	30	2	5	2154.0	127.4	21.60
20	30	2	8	2163.6	126.8	25.20
20	30	5	2	2176.8	126.1	29.76
20	30	5	5	2185.2	124.3	33.48
20	30	5	8	2198.4	124.1	37.20
20	30	8	2	2211.6	123.9	39.48
20	30	8	5	2232.0	121.4	42.00
20	30	8	8	2245.2	121.7	49.20
20	60	2	2	2954.4	150.4	15.60
20	60	2	5	2820.0	144.9	18.00
20	60	2	8	2772.0	140.8	21.60
20	60	5	2	2748.0	138.6	24.24
20	60	5	5	2832.0	140.8	28.80
20	60	5	8	2904.0	143.5	32.40
20	60	8	2	3022.8	146.0	35.16
20	60	8	5	3036.0	144.9	39.60
20	60	8	8	3056.4	143.8	44.88
50	0	2	2	3583.2	181.3	11.28
50	0	2	5	3601.2	180.8	14.40
50	0	2	8	3608.4	179.4	16.80
50	0	5	2	3616.8	179.1	21.12
50	0	5	5	3622.8	178.0	22.80
50	0	5	8	3637.2	177.6	27.60
50	0	8	2	3651.6	176.9	30.84
50	0	8	5	3672.0	175.3	36.00
50	0	8	8	36852	174.7	40.56
50	30	2	2	1195.2	123.6	52.80
50	30	2	5	1212.0	118.7	60.00
50	30	2	8	1236.0	115.9	64.80
50	30	5	2	1251.6	115.1	68.64
50	30	5	5	1272.0	113.2	75.60
50	30	5	8	1296.0	111.8	82.80
50	30	8	2	1318.8	110.7	88.08
50	30	8	5	1344.0	108.2	97.20
50	30	8	8	1388.4	106.3	107.64
50	60	2	2	2176.8	132.8	40.56
50	60	2	5	2196.0	131.1	45.60
50	60	2	8	2220.0	129.7	52.80
50	60	5	2	2245.2	128.3	60.00
50	60	5	5	2286.0	127.0	67.20
50	60	5	8	2294.4	125.6	73.20
50	60	8	2	2313.6	123.9	79.44
50	60	8	5	2340.0	114.5	85.20
50	60	8	8	2382.0	119.5	99.00
80	0	2	2	2988.0	154.8	31.92
80	0	2	5	3012.0	153.2	36.00
80	0	2	8	3036.0	151.8	43.20
80	0	5	2	3056.4	150.4	51.36
80	0	5	5	3108.0	150.7	61.20
80	0	5	8	3156.0	151.2	72.00

80	0	8	2	3244.8	151.5	82.80
80	0	8	5	3228.0	144.9	86.40
80	0	8	8	3193.2	140.8	90.36
80	30	2	2	3616.8	185.7	23.28
80	30	2	5	3639.6	183.5	30.00
80	30	2	8	3660.0	182.2	36.00
80	30	5	2	3685.2	181.3	42.72
80	30	5	5	3708.0	179.4	48.00
80	30	5	8	3732.0	178.0	54.00
80	30	8	2	3753.6	176.9	62.16
80	30	8	5	3672.0	175.3	73.20
80	30	8	8	3822.0	172.5	81.72
80	60	2	2	1218.0	128.3	87.00
80	60	2	5	1248.0	125.6	94.80
80	60	2	8	1272.0	124.2	103.20
80	60	5	2	1318.8	121.7	116.16
80	60	5	5	1344.0	118.7	126.00
80	60	5	8	1392.0	115.9	136.80
80	60	8	2	1422.0	115.1	145.44
80	60	8	5	1464.0	110.4	156.00
80	60	8	8	1524.0	108.5	174.72
$\min y_i(X), i = 1,\dots,81$				1149.6	92.4	11.3
$\max y_i(X), i = 1,\dots,81$				3822.0	161.5	174.7

In the decision, it is desirable to obtain the value of the score for the first and third characteristics (criterion) as high as possible: : $h_1(Y) \to max,\ h_3(Y) \to max$; second and fourth as low as possible: $h_2(Y) \to min,\ h_4(Y) \to min.$

Parametrical restrictions change in the following limits:

$$y_1 \in [20.50.80.], y_2 \in [0.30.60.], y_3 \in [2.0\ 5.0\ 8.0], y_4 \in [2.2\ 5.5\ 8.8]. \quad (6.3)$$

$21 \le y_1 \le 79,\ 5 \le y_2 \le 59,\ 2.1 \le y_3 \le 9,\ 2.2 \le y_4 \le 7.0.$

The chemical composition of the material of the product is determined (per unit volume, weight) by the percentage content of a certain set of material components, which in total are equal to one hundred percent:

$$y_1 + y_2 + y_3, + y_4 = 100. \quad (6.4)$$

It is required. 1) To construct mathematical model of structure of the studied material in the form of a vector problem of mathematical programming.

2) To carry out model operation: first, on the basis of the constructed mathematical model, secondly, on the basis of methods of solution of a vector problem of non-linear programming at equivalent criteria, and, thirdly, the software developed for these purposes in the MATLAB system.

3) To make an optimal solution: The choice of optimum composition (structure) of material according to its functional characteristics taking into account their equivalence.

4) To choose the optimum composition of structure of material according to its functional characteristics taking into account a priority of the third criterion.

Stage 1a. Construction of a mathematical model of the structure of the material in conditions of certainty and uncertainty in a general form. The construction of a mathematical model for making an optimal management decision on the structure of the material is shown in section 2.3. In accordance with (2.21)-(2.27), we will present a mathematical model of the material under conditions of certainty in the form of a vector optimization problem:

$$Opt\ H(Y) = \{\max H_1(Y) = \{\max h_k(Y), k = \overline{1, K_1^{def}}\}, \quad (6.5)$$

$$\min H_2(Y) = \{\min h_k(Y), k = 1, K_2^{def}, \quad (6.6)$$

$$\text{at restrictions } G(Y) \le B,\ \sum_{v=1}^{V} y_v(t) = 100\%, \quad (6.7)$$

$$h_k^{min} \le h_k(Y) \le h_k^{max}, k = \overline{1, K}, y_j^{min} \le y_j \le y_j^{max}, j = \overline{1, N}, \quad (6.8)$$

where $Y = \{y_j,\ j = \overline{1, N}\}$ is a vector of controlled variables (constructive parameters);

$H(Y) = \{H_1(Y)\ H_2(Y)\}$ - vector criterion, each component of which represents a vector of criteria (characteristics) of the material, which functionally depend on Y values of the vector of variables;

in (6.8) $h_k^{min} \le h_k(Y) \le h_k^{max}, k = \overline{1,K}$ is a vector-function of the constraints imposed on the functioning of the material;

in (6.8) $y_j^{min} \le y_j \le y_j^{max}, j = \overline{1,N}$ are parametric constraints.

It is assumed that the functions $h_k(Y), k = \overline{1,K}$ are differentiable and convex, $g_i(Y),\ i = \overline{1,M}$ are continuous, and the set of valid points $\boldsymbol{S}$ given by constraints (6.8) is not empty and is a compact:

$\boldsymbol{S} = \{Y \in R^n | G(Y) \le 0, Y^{min} \le Y \le Y^{max}\} \ne \emptyset.$

6.1.2. Stage 2. ***Construction of a numerical model of the structure of a material under conditions of certainty***

The construction of a model of the structure of a material under conditions of certainty is determined by the functional dependence of each characteristic, restrictions on the parameters of the material. In our example, the characteristic (6.2) and the constraints (6.1) are known. Using data (6.1), (6.2), we will build a numerical model in the form of a vector problem of nonlinear programming (6.5)-(6.8) under conditions of certainty:

$$max\ h_1(Y) = 323.84 - 2.249y_1 - 3.49y_2 + 10.7267y_3 + 13.124y_4 + 0.0968y_1y_2 - 0.062y_1y_3 - 0.169y_1y_4 + 0.0743y_2y_3 - 0.1042y_2y_4 - 0.0036y_3y_4 + 0.0143y_1^2 + 0.0118y_2^2 - 0.2434y_3^2 - 0.5026y_4^2, \quad (6.9)$$

$$\text{at restrictions: } y_1 + y_2 + y_3 + y_4 = 100, \quad (6.10)$$

$$21 \le y_1 \le 79, 5 \le y_2 \le 59, 2.1 \le y_3 \le 9.0, 2.2 \le y_4 \le 7.0. \quad (6.11)$$

These data are further used in the construction of a general mathematical model of the material (under conditions of certainty and uncertainty).

6.1.3. Stage 3.1. ***Transformation of experimental data (uncertainty conditions) into data with functional dependence (certainty conditions) and construction of a numerical model***

Conditions of uncertainty are characterized by the fact that the initial data characterizing the object under study are represented by: a) random, b) fuzzy, or c) incomplete data, i.e., under conditions of uncertainty, only a finite set of measured parameters $= \overline{1,Y}$ are known:

$Y_v = \{y_{iv}, v = \overline{1,V}\}, i = \overline{1,M}$, where $v = \overline{1,V}$ is the number of components (parameters) from which the material can be composed (manufactured), $i = \overline{1,M}$ is the number and set of data; and the corresponding set of K characteristics:

$h_k(Y_v = \{y_{iv},\ v = \overline{1,V}\}, i = \overline{1,M}), k = \overline{1,K}.$

Therefore, *under conditions of uncertainty*, there is not enough information about the functional dependence of each characteristic and constraints on the parameters. The information data of options a) and b) shall be converted into numerical data of option c) and shall be presented in tabular form. The paper considers option c) information with incomplete data, which, as a rule, ***is obtained as a result of an experiment.***

Taking into account the measured parameters, Y_v and the corresponding set of K characteristics: $h_k(Y_v = \{y_{iv},\ v = \overline{1,V}\}, i = \overline{1,M}), k = \overline{1,K}$ Let us present a matrix of results of experimental data on the material under study:

$$I = \begin{vmatrix} a_1 \\ \dots \\ a_M \end{vmatrix} = \begin{vmatrix} Y_1 = y_{11}, y_{12}, y_{13}, y_{14}\ h_2(Y_1), h_3(Y_1), h_4(Y_1) \\ \dots \\ Y_M = y_{M1}, y_{M2}, y_{M3}, y_{M4} h_2(Y_M), h_3(Y_M), h_4(Y_M) \end{vmatrix}, \quad (6.12)$$

Let us present a mathematical model of the structure of the material under uncertainty in the form of a vector problem of mathematical programming:

$$Opt\ H(X) = \{\max I_1(Y) \equiv \{\max h_k\ (Y_i,\ i = \overline{1,M})\}^T, k = \overline{1,K_1^{unc}}\}, \quad (6.13)$$

$$\min I_2(Y) \equiv \{\min h_k\ (Y_i,\ i = \overline{1,M})\}^T, k = \overline{1,K_2^{unc}}\}\}, \quad (6.14)$$

$$\text{at restriction} \quad h_k^{min} \le h_k(Y) \le h_k^{max}, k = \overline{1,K}, \quad (6.15)$$

$$\textstyle\sum_{v=1}^{V} y_v(t) = 100\%, y_v^{min} \le y_v \le y_v^{max}, v = \overline{1,V}, \quad (6.16)$$

where $Y = \{y_v,\ v = \overline{1,V}\}$ is the vector of controlled variables (parameters);

$H(Y) = \{I_1(Y)\ I_2(Y)\}$ is a vector criterion, each component of which represents a vector of criteria (output characteristics of the object under study). The value of the characteristic (function) depends on the discrete values of the vector of the variables Y. $I_1(Y)=\overline{1,K_1^{unc}}$, $I_2(Y)=\overline{1,K_2^{unc}}$ (uncertainty) – a set of max and min criteria formed under uncertainty; in (6.15) $h_k^{min} \le h(X) \le h_k^{max}, k=\overline{1,K}$ – vector-function of the restrictions imposed on the functioning of the object under study, $y_v^{min} \le y_v \le y_v^{max}$, $v=\overline{1,V}$ – parametric constraints of the object under study.

6.1.3. Stage 3.2. ***Construction of a numerical model of the structure of a material under uncertainty***

The construction of a numerical model of the structure of the material under uncertainty consists in the use of qualitative and quantitative descriptions of the material, the experimental data obtained on the principle of "input-output" in Table 1..

Transformation of information (initial data in Table 1):

$h_2(Y_i, i=\overline{1,M}), h_3(Y_i, i=\overline{1,M}), h_4(Y_i, i=\overline{1,M})$ into a functional form: $h_2(Y), h_3(Y), h_4(Y)$ is carried out by using mathematical methods (regression analysis). The initial data of Table 2 are formed in the MATLAB system in the form of a matrix:

$$I=|Y,H| = \{y_{i1}, y_{i2}, y_{i3}, y_{i4}, h_{i2},\ h_{i3},\ h_{i4}, i=\overline{1,M}\}. \quad (6.17)$$

Для каждого набора экспериментального данных $h_k, k=2,3,4$ строится функция регрессии методом наименьших квадратов $\mathbf{min}\sum_{i=1}^{M}(y_i-\overline{y_i})^2$ в системе MATLAB. Для этого формируется полином A_k, определяющий взаимосвязь параметров: $Y_i=\{y_{i1}, y_{i2}, y_{i3}, y_{i4}\}$ и функции $\overline{y_{ki}}=h(Y_i, A_k), k= 2,3,4$. Результатом является система коэффициентов: $A_k=\{A_{0k}, A_{1k}, \dots, A_{14k}\}$, которые определяют коэффициенты квадратичного полинома:

For each set of experimental data $h_k, k=2,3,4$ a regression function is constructed by the method of least squares $\mathbf{min}\sum_{i=1}^{M}(y_i-\overline{y_i})^2$ in the MATLAB system. To do this, a polynomial A_k is formed, which defines the relationship of the parameters: $Y_i=\{y_{i1}, y_{i2}, y_{i3}, y_{i4}\}$ and the functions of

$\overline{y_{ki}}=h(Y_i, A_k), k= 2,3,4$. The result is a system of coefficients: $Y_i=\{y_{i1}, y_{i2}, y_{i3}, y_{i4}\}$, which determine the coefficients of the quadratic polynomial:

$$h_k(Y,A)=A_{0k}+A_{1k}y_1+A_{2k}y_2+A_{3k}y_3+A_{4k}y_4+A_{5k}y_1y_2+A_{6k}y_1y_3+A_{7k}y_1y_4+A_{8k}y_2y_3+A_{9k}y_2y_4+A_{10k}y_3y_4+A_{11k}y_1^2+A_{12k}y_2^2+A_{13k}y_3^2+A_{14k}y_4^2, k= 2,3,4. \quad (6.18)$$

Polynomial approximation software with four variables and fourteen factors is presented in [44]. As a result, the experimental data of Table 1 are transformed into a system of coefficients of three functions of the form (6.18) in the form of a table (Program Z_Material_MMTT32_os13_4k).

Polynomial approximation software with four variables and fourteen factors is presented in [44]. As a result, the experimental data of Table 1 are transformed into a system of coefficients of three functions of the form (4.18) in the form of a table (Program: Z_Material_MMTT32_os13_4k):

```
Ao=[323.8408  954.8634  110.02   21.0051     % A0k      (6.19)
    -2.2495   28.6719   0.9106  -0.0101      %A1k
    -3.4938   37.0392   0.6206  -0.8403      %A2k
    10.7267  -31.0303  -0.4287  -0.4314      %A3k
    13.1239  -54.0031  -2.5176   1.1718      %A4k
     0.0969   -0.9219  -0.0151   0.0166      %A5k
    -0.0621    0.5644  -0.0094   0.0850      %A6k
    -0.1696    0.8966   0.0222  -0.0001      %A7k
     0.0743   -0.1540  -0.0198   0.0522      %A8k
    -0.1042    0.3919   0.0184   0.0003      %A9k
     0.0036   -0.0135  -0.0006   0.0006      %A10k
     0.0142    0.0477  -0.0004  -0.0021      %A11k
     0.0117    0.0437  -0.0003   0.0035      %A12k
    -0.2433    3.8489   0.0390   0.0061      %A13k
    -0.5026    3.1748   0.1414  -0.0310];    %A14k
Rj =[0.6115   0.7149   0.6551   0.9017];
RRj=[0.3740   0.5111   0.4292   0.8130];
```

На основе Ao(2) Ao(3) Ao(4) строятся функции $h_2(Y)$, $h_3(Y)$ и $h_4(Y)$, которые с учетом полученных коэффициентов (4.19) являются ***численной моделью структуры материала в условиях неопределенности***:

On the basis of Ao(2) Ao(3) Ao(4), the functions $h_2(Y)$, $h_3(Y)$ and $h_4(Y)$ are constructed, which, taking into account the obtained coefficients (4.19), are ***a numerical model of the structure of the material under uncertainty***:

$$Opt\ H(Y) = \{max\ h_3(Y) \equiv 110.22 + 0.7918y_1 + 1.73y_2 - 0.3713y_3 - 2.20y_4 - 0.0132y_1y_2 - 0.008y_1y_3 + 0.0193y_1y_4 - 0.0172y_2y_3 + 0.0161y_2y_4 - 0.0006y_3y_4 - 0.0004y_1^2 - 0.0002y_2^2 + 0.0335y_3^2 + 0.124y_4^2, \quad (6.20)$$

$$min\ h_2(Y) \equiv 954.86 + 28.67y_1 + 37.03y_2 - 31.03y_3 + 54y_4 - 0.922y_1y_2 - 2y_1y_3 + 0.896y_1y_4 - 0.154y_2y_3 + 0.3919y_2y_4 - 0.0134y_3y_4 + 0.0478y_1^2 + 0.0438y_2^2 + 3.8489y_3^2 + 3.1748y_4^2, \quad (6.21)$$

$$min\ h_4(Y) \equiv 21.004 - 0.0097y_1 - 0.841y_2 - 0.4326y_3 + 1.1723y_4 + 0166y_1y_2 + 0.085y_1y_3 - 0.0001y_1y_4 + 0.0523y_2y_3 + 0.0002y_2y_4 + 0.0006y_3y_4 - 0.0022y_1^2 + 0.0035y_2^2 + 0.006y_3^2 - 0.0311y_4^2\}, \quad (6.22)$$

at restriction: $y_1 + y_2 + y_3 + y_4 = 100,$ (6.23)

$$21 \le y_1 \le 79, 5 \le y_2 \le 59, 2.1 \le y_3 \le 9.0, 2.2 \le y_4 \le 7.0. \quad (6.24)$$

Minimum and maximum values of experimental data are $y_1, \dots, y_4$ are presented at the bottom of Table 2. The minimum and maximum values of the functions $h_1(Y), h_3(Y), h_2(Y), h_4(Y)$ differ slightly from the experimental data. The correlation index and coefficients of determination are presented in the lower rows of Table 2. The results of regression analysis (6.20)-(6.24) are further used when it is necessary to build a general mathematical model of the material.

6.1.4. Stage 4. *Construction of an aggregated mathematical and numerical model of the structure of the material under conditions of certainty*

Combining mathematical models of the structure of the material under conditions of certainty (6.5)-(6.8) and uncertainty (6.13)-(6.16), we will present *a mathematical model of the material under conditions of certainty and uncertainty* in the aggregate in the form of a vector problem:

$$Opt\ H(Y) = \{\max H_1(Y) = \{\max h_k(Y), k = \overline{1, K_1^{def}}\}, \quad (6.25)$$

$$\max I_1(Y) \equiv \{\max h_k(Y_i,\ i = \overline{1, M})\}^T, k = \overline{1, K_1^{unc}}\}, \quad (6.26)$$

$$\min H_2(Y) = \{\min h_k(X), k = \overline{1, K_2^{def}}, \quad (6.27)$$

$$\min I_2(Y) \equiv \{\min h_k(Y_i,\ i = \overline{1, M})\}^T, k = \overline{1, K_1^{unc}}\}\}, \quad (6.28)$$

at restriction: $h_k^{min} \le h_k(Y) \le h_k^{max}, k = \overline{1, K}, y_j^{min} \le y_j \le y_j^{max}, j = \overline{1, N},$ (6.29)

where $Y = \{y_j,\ j = \overline{1, N}\}$ - vector of controlled variables (design parameters);

$H_1(Y) = \{h_k(Y), k = \overline{1, K_1^{def}}\}$, $H_2(Y) = \{h_k(Y), k = \overline{1, K_2^{def}}\}$ –

many max and min functions, respectively;

$I_1(Y) = \{\{h_k(Y_i, i = \overline{1, M})\}^T, k = \overline{1, K_1^{unc}}\}_{\overline{1,N}}$, $I_2(Y) = \{\{h_k(Y_i, i = \overline{1, M})\}^T, k = \overline{1, K_2^{unc}}\}$ - Multiple max and min matrices, respectively; (definiteness), $K_1^{unc}.K_2^{unc}$ (uncertainty) is a set of criteria max and min formed under conditions of certainty and uncertainty;

Combining numerical models of the structure of the material under conditions of certainty (6.9)-(6.11) and uncertainty (6.20)-(6.24), we will present *a numerical model of the material under conditions of certainty and uncertainty* in the aggregate in the form of a vector problem:

$$opt\ F(X) = \{\max H_1(Y) = \{\max h_1(X) \equiv 323.84 - 2.25y_1 - 3.49y_2 + 10.72y_3 + 13.124y_4 + 0.0968y_1y_2 - 0.062y_1y_3 - 0.169y_1y_4 + 0.0743y_2y_3 - 0.1y_2y_4 - 0.0036y_3y_4 + 0.0143y_1^2 + 0.0118y_2^2 - 0.2434y_3^2 - 0.5026y_4^2, \quad (6.30)$$

$$max\ h_3(Y) \equiv 110.22 + 0.7918y_1 + 1.73y_2 - 0.3713y_3 - 2.20y_4 - 0.0132y_1y_2 - 0.008y_1y_3 + 0.0193y_1y_4 - 0.0172y_2y_3 + 0.0161y_2y_4 - 0.0006y_3y_4 - 0.0004y_1^2 - 0.0002y_2^2 + 0.0335y_3^2 + 0.124y_4^2\}, \quad (6.31)$$

$$\min H_2(Y) = \{min\ h_2(Y) \equiv 954.86 + 28.67y_1 + 37.03y_2 - 31.03y_3 + 54y_4 - 0.922y_1y_2 - 2y_1y_3 + 0.896y_1y_4 - 0.154y_2y_3 + 0.3919y_2y_4 - 0.0134y_3y_4 + 0.0478y_1^2 + 0.0438y_2^2 + 3.8489y_3^2 + 3.1748y_4^2, \quad (6.32)$$

$$max\ h_4(Y) \equiv 21.004 - 0.0097y_1 - 0.841y_2 - 0.4326y_3 + 1.1723y_4 + 0166y_1y_2 + 0.085y_1y_3 - 0.0001y_1y_4 + 0.0523y_2y_3 + 0.0002y_2y_4 + 0.0006y_3y_4 - 0.0022y_1^2 + 0.0035y_2^2 + 0.006y_3^2 - 0.0311y_4^2\}\}, \quad (6.33)$$

at restrictions: $y_1 + y_2 + y_3 + y_4 = 100,$ (6.34)

$$21 \le y_1 \le 79, 5 \le y_2 \le 59, 2.1 \le y_3 \le 9.0, 2.2 \le y_4 \le 7.0. \quad (6.35)$$

The vector problem of mathematical programming (6.30)-(6.35) is a numerical model for making an optimal decision of the structure of a material under conditions of certainty and uncertainty in the aggregate.

6.2. Block 2. Methodology of the process of making an optimal decision (selection of optimal parameters) of the material structure based on the vector problem (VPMP

6.2.1. Stage 5. *Solution of VPMP - model of material structure with equivalent criteria (solution of a direct problem).*

For solving vector problems of mathematical programming (6.30)-(6.35), methods based on axiomatics and the principle of optimality 1 are presented. The algorithm is presented as a series of steps.

Step 1. Decides problem (6.30) - (6.35) by each criterion separately, at the same time the function *fmincon*(…) of the MATLAB system is used, the appeal to the function *fmincon*(…) is considered in [44]. As a result of calculation for each criterion, we receive optimum points:

X_k^* and $f_k^* = f_k(X_k^*), k = \overline{1,K}, \boldsymbol{K} = 4$

sizes of criteria in this point, i.e. the best decision on each criterion:

1: $Y_1^* = \{y_1 = 46.56, y_2 = 43.23, y_3 = 8.0, y_4 = 2.2\}, h_1^* = h_1(Y_1^*) = -387.9;$

2: $Y_2^* = \{y_1 = 55.60, y_2 = 34.19, y_3 = 8.0, y_4 = 2.2\}, h_2^* = h_2(Y_2^*) = 1361.4;$

3: $Y_3^* = \{y_1 = 31.90, y_2 = 59.00, y_3 = 2.1, y_4 = 7.0\}, h_3^* = h_3(Y_3^*) = -210.3;$

4: $Y_4^* = \{y_1 = 36.70, y_2 = 59.00, y_3 = 2.1, y_4 = 2.2\}, h_4^* = h_4(Y_4^*) = 30.714$

The result the solution of a problem of non-linear programming (6.30)-(6.35) in three-dimensional frames of x_1, x_3 and $f_1(X)$, $f_2(X)$, $f_3(X)$, $f_4(X)$ is presented on Fig 6.5, 6.6, 6.7, 6.8.

The location of the optimum points $X_1^*, X_2^*, X_3^*, X_4^*$ in the region of the constraints (6.30)-(6.35) in the coordinates $\{x_1, x_3\}$ is shown in Figure 6.1. The set of points of S^o lying in the domain of restrictions between the points $X_1^*, X_2^*, X_3^*, X_4^*$ represent a set of Pareto optimal points.

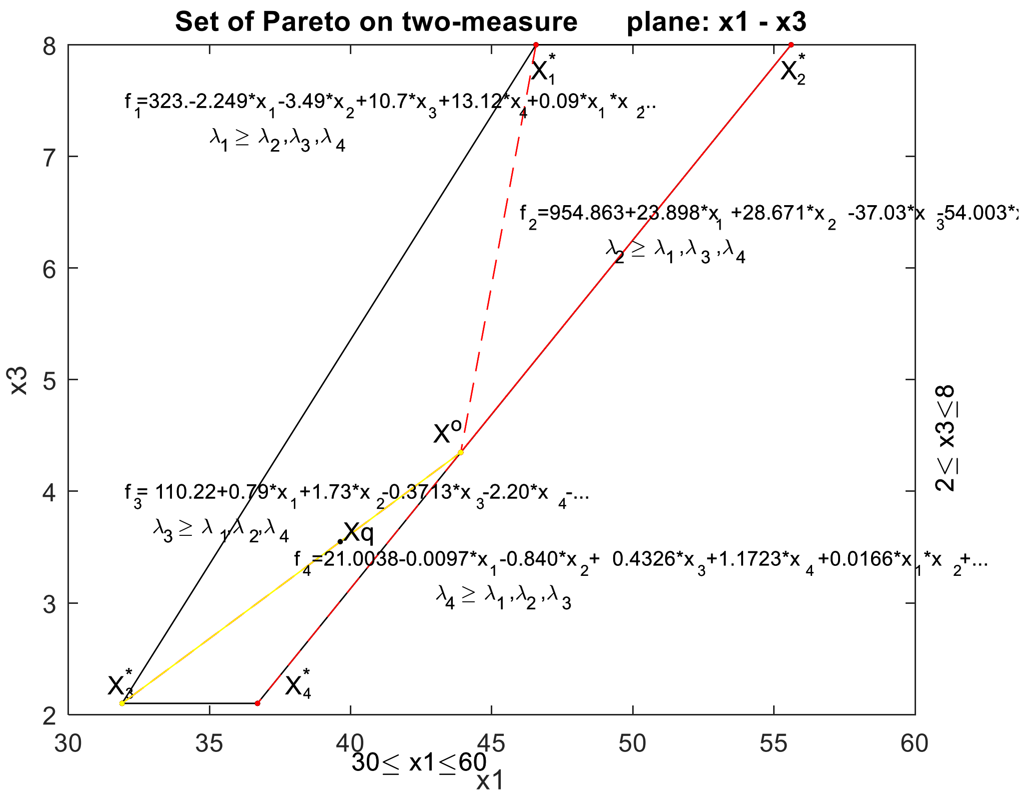

Figure 6.1. The set of admissible points and Pareto optimal $\boldsymbol{S^o} \subset \boldsymbol{S}$, $X_1^*, X_2^*, X_3^*, X_4^*$ in the coordinate system $\{x_1, x_3\}$

Step 2. Determine the worst value of each criterion (antioptimum): Y_k^0 and $h_k^0 = h_k(Y_k^0), k = \overline{1,K}$, $\boldsymbol{K}$=4. Why is problem (6.30)-(6.35) solved for each criterion $k = \overline{1,K_1}$ by the minimum, for each criterion $k = \overline{1,K_2}$ to the maximum. As a result of the solution, we get: $X_k^0 = \{x_j, j = \overline{1,N}\}$ - the optimum point for the corresponding criterion, $k = \overline{1,K}$; $f_k^0 = f_k(X_k^0)$ is the value of the k-th criterion at the point, $X_k^0, k = \overline{1,K}$ (superscript zero):

$Y_1^0 = \{y_1 = 31.9, y_2 = 59.0, y_3 = 2.1, y_4 = 7.00\}, h_1^0 = h_1(Y_1^0) = 296.6;$

$Y_2^0 = \{y_1 = 31.9, y_2 = 59.0, y_3 = 2.1, y_4 = 7.\}, h_2^0 = h_2(Y_2^0) = -2458.5;$

$Y_3^0 = \{y_1 = 78.16, y_2 = 9.02, y_3 = 8, y_4 = 4.81\}, h_3^0 = h_3(Y_3^0) = 169.26;$

$Y_4^0 = \{y_1 = 62.71, y_2 = 22.9, y_3 = 8, y_4 = 6.39\}, h_4^0 = h_4(Y_4^0) = -73.62.$

The obtained points of the anti-optimum $X_1^0, X_2^0, X_3^0, X_4^0$ are shown in Figures 6.5, ..., Figure 6.8 respectively.

Step 3. Systems analysis of a set of points that are Pareto-optimal is performed, (i.e. the analysis by each criterion). In points of an optimum of $Y^* = \{Y_1^*, Y_2^*, Y_3^*, Y_4^*\}$sizes of target functions of

$F(X^*) = \|f_q(X_k^*)\|_{q=\overline{1,K}}^{k=\overline{1,K}}$,

a vector of $D = (d_1\ d_2\ d_3\ d_4)^T$ of deviations are determined by each criterion on an admissible set of $\boldsymbol{S}$: $d_k = h_k^* - h_k^0, k = \overline{1,4}$, and matrix of the relative estimates of

$/\lambda(Y^*) = \|\lambda_q(Y_k^*)\|_{q=\overline{1,K}}^{k=\overline{1,K}}$, where $\lambda_k(X) = (h_k^* - h_k^0)/d_k$

$$H(Y^*) = \left\|\begin{matrix} 388.0\ 1444.2\ 183.9\ 68.5 \\ 382.0\ 1361.4\ 177.3\ 72.1 \\ 296.6\ 2458.5\ 210.4\ 30.2 \\ 330.1\ 2210.9\ 208.0\ 30.7 \end{matrix}\right\|, d_k = \left\|\begin{matrix} 91.4 \\ -1097 \\ 41.09 \\ -42.9 \end{matrix}\right\|,$$

$$\lambda(Y^*) = \left\|\begin{matrix} 1.0000\ 0.9245\ 0.3560\ 0.1197 \\ 0.9367\ 1.0000\ 0.1968\ 0.0363 \\ 0.0 \quad\quad 0.0\ 1.0000\ 1.011 \\ 0.3669\ 0.2257\ 0.9427\ 1.0000 \end{matrix}\right\|. \quad (6.36)$$

The analysis of sizes of criteria in the relative estimates shows that at the points of the optimum $Y^* = \{Y_1^*, Y_2^*, Y_3^*, Y_4^*\}$ (on diagonal) the relative assessment is equal to unit. Other criteria there is much less unit. It is required to find such point (parameters) at which the relative estimates are closest to unit. The solution of this problem is directed to the solution of λ-problem - step 4, 5.

Step 4. Creation of the λ-problem is carried out in two stages: originally the maximine problem of optimization with the normalized criteria is under construction:

$$\lambda^o = \boldsymbol{max}_{X\in S}\boldsymbol{min}_{k\in K}\lambda_k(X), G(X)\le 0, X \ge 0, \quad (6.36)$$

which at the second stage will be transformed to a reference problem of mathematical programming (λ-problem):

$$\lambda^o = max\,\lambda, \quad (6.37)$$

at restrictions: $\lambda - \frac{h_1(Y)-h_1^0}{h_1^*-h_1^0} \le 0,\ \lambda - \frac{h_3(Y)-h_3^0}{h_3^*-h_3^0} \le 0,$ (6.38)

$$\lambda - \frac{h_2(Y)-h_2^0}{h_2^*-h_2^0} \le 0,\ \lambda - \frac{h_4(Y)-h_4^0}{h_4^*-h_4^0} \le 0, \quad (6.39)$$

$$y_1 + y_2 + y_3 + y_4 = 100, \quad (6.40)$$

$$0\le\lambda\le 1,\quad 21 \le y_1 \le 79, 5 \le y_2 \le 59, 2.1 \le y_3 \le 9.0, 2.2 \le y_4 \le 7.0. \quad (6.41)$$

where the vector of unknowns has dimension of $N + 1$: $\boldsymbol{Y} = \{y_1, \dots, y_N, \lambda\}$;

functions $h_1(Y), h_2(Y), h_3(Y), h_4(Y)$. correspond to (6.30)-(6.35). Substituting the numerical values of the functions $h_1(Y), h_2(Y), h_3(Y), h_4(Y)$, we get λ-problem:

$$\lambda^o = max\,\lambda, \quad (6.42)$$

at restrictions: $\lambda - \frac{323.84-2.249y_1-3.49*x2\ \dots -0.2434y_3^2-0.5026y_4^2-h_1^0}{h_1^*-h_1^0} \le 0,$ (6.43)

$$\lambda - \frac{110.22+0.7918y_1+1.73y_2-\dots+0.0335y_3^2+0.124y_4^2-h_3^0}{f_3^*-f_3^0} \le 0, \quad (6.44)$$

$$\lambda - \frac{954.8+28.67y_1+37y_2-\dots+3.8489y_3^2+3.1748y_4^2-h_2^0}{h_2^*-h_2^0} \le 0, \quad (6.45)$$

$$\lambda - \frac{21-0.0097y_1-0.841y_2-\dots+0.006y_3^2-0.0311y_4^2-h_4^0}{h_4^*-h_4^0} \le 0, \quad (6.46)$$

$$y_1 + y_2 + y_3 + y_4 = 100, \quad (6.47)$$

$$0\le\lambda\le 1,\quad 21 \le y_1 \le 79, 5 \le y_2 \le 59, 2.1 \le y_3 \le 9.0, 2.2 \le y_4 \le 7.0. \quad (6.48)$$

Step 5. Solution of the λ-problem.

For this purpose we use the function *fmincon*(…): [*Xo*,*Lo*]=fmincon(*'Z_TehnSist_4Krit_L'*,*X0*,*Ao*,*bo*,*Aeq*,*beq*,*lbo*,*ubo*,*'Z_TehnSist_LConst'*,*options*).

As a result of the solution of VPMP (6.30)-(6.35) at equivalent criteria and λ-problem corresponding to it (6.42)-(6.48) received:

$$\boldsymbol{X^o} = \{\boldsymbol{Y^o} = \{y_1 = 43.9, y_2 = 49.54, y_3 = 4.348, y_4 = 2.2, \lambda^o = 0.6087\}, \quad (6.49)$$

an optimum point – design data of material, $\boldsymbol{X^o}$.

The optimum point $\boldsymbol{X^o}$, which represents the design parameters of the material under equivalent criteria (characteristics, shown in Fig. 6.1; $h_k(Y^o), k = \overline{1,K}$ - values of criteria (characteristics of the structure of the material):

$$\{h_1(Y^o) = 364.0, h_2(Y^o) = 1790.7, h_3(Y^o) = 194.3, h_4(Y^o) = 47.5\}; \quad (6.50)$$

$\lambda_k(Y^o), k = \overline{1,K}$- values of relative estimates

$$\{\lambda_1(Y^o) = 0.7372, \lambda_2(Y^o) = 0.6087, \lambda_3(Y^o) = 0.6087, \lambda_4(Y^o) = 0.6087\}; \quad (6.51)$$

λ^o=0.6087 is the highest lower level among all relative estimates, measured in relative units::

$\lambda^o = min\,(\lambda_1(Y^o), \lambda_2(Y^o), \lambda_3(Y^o), \lambda_4(Y^o)) = 0.6087.$

λ^o – also called the guaranteed result in relative units. The guaranteed result λ^o shows that λ^o=0.6087 and the characteristics of the material in relative units $\lambda_k(Y^o)$ and, accordingly, the characteristics of the structure of the material $h_k(Y^o)$ cannot be improved without degrading other characteristics.

Note that according to the theorem 1, at the point Y^o Criteria 2, 3 and 4 are contradictory. This contradiction is determined by the equality of:

$\lambda_2(Y^o) = \lambda_3(Y^o) = \lambda_4(Y^o) = \lambda^o = 0.6087,$

and the rest of the criteria are inequality$\{\lambda_2(X^o) = 0.7372\} > \lambda^o$.

Theorem 1 serves as the basis for determining the correctness of the solution of the vector problem. In a vector problem of mathematical programming, as a rule, the equation for two criteria is satisfied:

$\lambda^o = \lambda_q(Y^o) = \lambda_p(Y^o),\ q, p \in \boldsymbol{K}, X \in S$, (in our example, such criteria are 2, 3, 4), for the other criteria it is defined as inequality.

6.2.2. Stage 6. *Geometric interpretation of the results of the VPMP solution with 4 parameters and 4 criteria into a two-dimensional coordinate system (with 2 parameters) in relative units.*

For a geometric interpretation of the results of the VPMP solution with 4 parameters and 4 criteria, we will introduce changes in the two-dimensional coordinate system (with 2 parameters) in relative units. The VPMP (6.30)-(6.35) parameters are y_1 and y_3 are considered as variables, parameters y_2 and y_4 are considered permanent. Let's assign a dimension to the constant parameters:

$y_2 = 49.5492, y_4 = 2.2$ according to the outcome of the VPMP decision (6.30)-(6.35) with equivalent criteria presented in (6.49). As a result, the VPMP (6.30)-(6.35) became two-dimensional. As a result of the decision of VPMP (6.30)-(6.35) with two variables y_1 and y_3 (additional "o" Y1omax was introduced into the designation of the results) obtained.

1. Coordinates of the point according to the first criterion to the maximum:

Y1omax = {x_1 =46.5676 x_2 =49.5492 x_3 =8.0000 x_4 =2.2000}. (6.52)

Values of the four criteria at the point Y1omax:

FY1omax = {$f_1(Y1omax) = 403.6$ $f_2(Y1omax) =$ 1430.3 $f_3(Y1omax)$ =190.2 $f_4(Y1omax)$ =72.7.

The values are relative to the estimates of the criteria at the point X1omax:

LY1omax = {$\lambda_1(Y1omax)$ =1.1708 $\lambda_2(Y1omax)$ = 0.9372
$\lambda_3(Y1omax)$ =0.5098 $\lambda_4(Y1omax)$ =0.0207. (6.53)

Coordinates of the point according to the first criterion for the minimum:

Y1omin ={31.9000 49.5492 2.1000 2.2000}.

The values of the six criteria at the point Y1omin are:

FY1omin ={1.0e+03 * 0.3037 2.2073 0.1957 0.0243}.

The values are relative to the estimates of the criteria at the point Y1omin:

LY1omin =(0.0774 0.2290 0.6441 1.1503) (6.54)

2. Coordinates of the point, functions and relative estimates for the second criterion for maximum and minimum:

Y2omax ={55.6075 49.5492 8.0000 2.2000}.
FY2omax ={1.0e+03 * 0.4320 1.1936 0.1909 0.0842}.
LY2omax ={1.4813 1.1530 0.5268 -0.2467}.
Y2omin ={31.9000 49.5492 2.1000 2.2000).
FY2omin ={1.0e+03 * 0.3037 2.2073 0.1957 0.0243}.
LY2omin ={0.0774 0.2290 0.6441 1.1503}. (6.55)

3. Coordinates of the point, functions and relative estimates for the third criterion for maximum and minimum:

Y3omax ={31.9000 49.5492 2.1000 2.2000 }.
FY3omax ={1.0e+03 * 0.3037 2.2073 0.1957 0.0243}.
LY3omax ={0.0774 0.2290 0.6441 1.1503}.
Y3omin ={78.1673 49.5492 8.0000 2.2000}.
FY3omin ={512.9622 636.7052 192.3963 111.3139}.
LY3omin ={2.3673 1.6606 0.5629 -0.8782}. (6.56)

4. Coordinates of the point, functions, and relative evaluations of the fourth criterion for maximum and minimum:

Y4omax ={36.7000 49.5492 2.1000 2.2000}. (6.57)
FY4omax ={1.0e+03 * 0.3182 2.1306 0.1964 0.0283}.
LY4omax ={0.2363 0.2989 0.6603 1.0562}.
Y4omin ={62.7123 49.5492 8.0000 2.2000}. (6.58)
FY4omin ={1.0e+03 * 0.4559 1.0129 0.1914 0.0930}.
LY4omin ={1.7432 1.3176 0.5391 -0.4511}. (6.59)

Let us present in general the results of the VPMP solution with two variable parameters x_1 and x_3 (two-dimensional VPMP):

Y =[Yopt(1,:)={46.5676 43.2324 8.0000 2.2000}, $\lambda_1(\text{Y1}omax)$ =0.5;
Yopt(2,:)= {55.6075 34.1925 8.0000 2.2000}, $\lambda_2(\text{Y2}omax)$ =0.6087;
Yopt(3,:)= {31.9000 59.0000 2.1000 7.0000}, $\lambda_3(\text{Y3}omax) = 0.6087$;
Yopt(4,:)= {36.7000 59.0000 2.1000 2.2000},$\lambda_4(\text{Y4}omax) = 0.7372$;
Yo(1:4)= {43.9022 49.5492 4.3486 2.200}, $\lambda(\text{Yo}) = \lambda^o = 0.5196$. (6.60)

In the admissible set of points $\boldsymbol{S}$ formed by constraints (6.47)-(6.48), the optimum points are Y_1^*, Y_2^*, Y, Y_4^* combined into a contour, represent a set of Pareto-optimal points, , $\boldsymbol{S^o} \subset \boldsymbol{S}$, ($\boldsymbol{S}$ are shown in Figure 6.1. The coordinates of these points, as well as the characteristics of the material in relative units $\lambda_1(Y), \lambda_2(Y), \lambda_3(Y), \lambda_4(Y)$ are shown in Figure 6.2 in three-dimensional space x_1 x_2 and λ, where the third axis λ is a relative estimate.

Discussion. Let's compare the results of the solution of VPMP (6.30)-(6.35) with the variable coordinates $\{y_1 y_2\ y_3\ y_4\}$ (four-dimensional VPMP) presented in (6.49), (6.50), (6.51), with the results of the VPMP solution (6.30)-(6.35) with variable coordinates $\{y_1\ y_3\}$ (two-dimensional VPMP) presented in (6.60). (In Figures 6.2, ..., 6.7, the vector $\boldsymbol{Y} = \{y_1, \dots, y_N, \lambda\}$ and the functions $h_1(Y), h_2(Y), h_3(Y), h_4(Y)$ are replaced by $\boldsymbol{X} = \{x_1, \dots, x_N, \lambda\}$; functions $f_1(X), f_2(X), f_3(X), f_4(X)$).

As a result of the comparison, we see that the values of the four functions $h_1(Y), h_2(Y), h_3(Y), h_4(Y)$ at the optimum point Y^o in the coordinates $\{y_1\ y_3\}$ and λ^o Match.

Optimal values of criteria $h_k(Y_k^*), k \in \boldsymbol{K}$ and the corresponding relative estimates do not match

Consider, for example, the optimal point, X_3^*. Function, $\lambda_3(X)$ is formed from the function $h_3(X)$ with variable coordinates $\{y_1\ y_3\}$ and with constant coordinates $\{y_2=49.54,\ y_4=2.2\}$, taken from the optimal point Y^o. (6.49). At the point Y_3^* the relative estimate is $\lambda_3(Y_3^*) = 0.6441$ – shown in Figure 6.2 with a black dot. But we know that the relative estimate of $\lambda_3(Y_3^*)$ derived from the function $h_3(Y_3^*)$ in the third step, it is equal to one, let's denote it as $\lambda_3^\Delta(Y_3^*) = 1$ – shown in Figure 6.2 with a red dot.

The difference between $\lambda_3^\Delta(Y_3^*) = 1$ and $\lambda_3(Y_3^*) = 0.6441$ $\lambda_3^\Delta(Y_3^*) = 1$ и $\lambda_3(Y_3^*) = 0.6441$ is an error Δ=0.3559 in the transition from a four-dimensional (and in the general case N-dimensional) to a two-dimensional system. The point X_1^* is shown in the same way. and the corresponding relative estimates, $\lambda_1(Y_1^*)$ and $\lambda_1^{\Delta}(Y_1^*)$. Summarizing and combining the problems of the discussion, we can formulate a methodology.

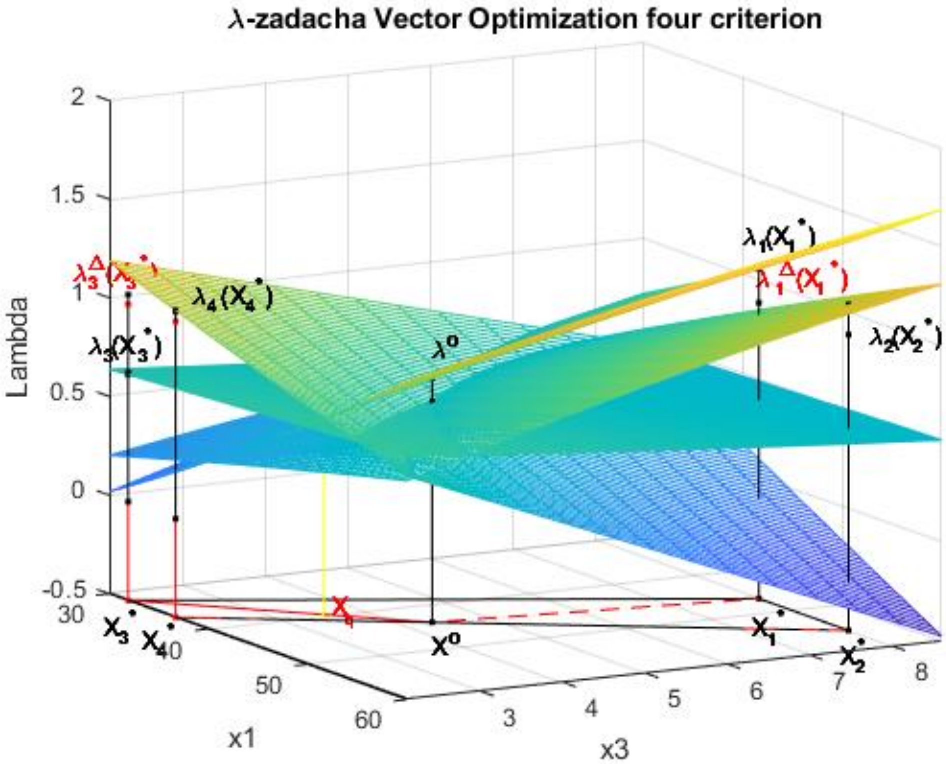

Figure 6.2. Geometric interpretation of the solution of the λ-problem $\lambda_1(X), \lambda_2(X), \lambda_3(X), \lambda_4(X)$ and X_1^*,X_2^*,X_3^*, X_4^* coordinate x_1 , x_3 $(= y_1\ y_3)$ and λ

Methodology of geometric interpretation of the transition from N-dimensional to two-dimensional dimension of function in a vector problem of mathematical programming.

Step 1. Construction and Solution of a λ-Problem with *N*-Dimensional Parameters.

Step 2. Construction and solution of a λ-problem with 2-dimensional parameters, the rest (N-2) parameters are constant taken from the results of solving a λ-problem with N-dimensional parameters (from step 1).

Step 3. Geometrical Construction of Functions from a λ-Problem with 2-Dimensional Parameters, Standard Methods, and Corresponding Labels.

Thus, for the first time in domestic and foreign practice, the transition and its geometric illustration from the *N*-dimensional to the two-dimensional dimension of the function in vector problems of mathematical programming with the corresponding approximation errors are shown.

6.2.3. Stage 7. *Decision-making in the structure of material model at the set priority of criterion (Algorithm 2. The solution of a vector task with a criterion priority).*

The person making decisions, as a rule, is the designer of material.

In the section, the variable *Y* has been replaced with the variable *X*.

Step 1. *The solution of a vector problem with equivalent criteria*. Results of the decision are presented in section 3.3. The numerical results of solving the vector problem are presented above.

A set of Pareto-optimal points $\boldsymbol{S^o} \subset \boldsymbol{S}$ is between the optimal points $X_1^*\ X^o\ X_3^*\ X^o\ X_4^*\ X^o\ X_2^*\ X^o X_1^*$ (in the figures *X* is used instead of *Y*). We will analyze the set of Pareto points $\boldsymbol{S^o} \subset \boldsymbol{S}$.

For this purpose, we will connect the auxiliary points:

$X_1^*\ X_3^*\ X_4^*\ X_2^*\ X_1^*$ with the point X^o, which conventionally represents the center of the Pareto set. As a result, four subsets of points $X \in \boldsymbol{S}_q^o \subset \boldsymbol{S^o} \subset \boldsymbol{S}, q = \overline{1,4}$. A subset of $\boldsymbol{S_1^o} \subset \boldsymbol{S^o} \subset \boldsymbol{S}$ (S is characterized by the fact that the relative estimate $\lambda_1 \geq \lambda_2, \lambda_3, \lambda_4$, i.e., in the field of the first criterion, $\boldsymbol{S_1^o}$. takes precedence over the others. Similar to $\boldsymbol{S_2^o}, \boldsymbol{S_3^o}, \boldsymbol{S_4^o}$ are subsets of points where the second, third, and fourth criteria take precedence over the others, respectively.

$\boldsymbol{S^o} = \boldsymbol{S_1^o} \cup \boldsymbol{S_2^o} \cup \boldsymbol{S_3^o} \cup \boldsymbol{S_4^o}$.

The coordinates of all the obtained points and the relative estimates are represented in the two-dimensional space The coordinates of all the obtained points and the relative estimates are represented in the two-dimensional space $\{x_1\ x_3\}$ in Figure 6.1. These coordinates are shown in three dimensioned spaces $\{x_1\ x_3\ \lambda\}$ in Figure 6.2, where the third axis λ is a relative estimate. The limitations of the set of Pareto-optimal points are reduced to -0.5 in Figure 6.2. This information is also the basis for further study of the structure of the Pareto set in Figure 6.1. The decision-maker, as a rule, is the developer of the system (material structure). If the results of solving a vector problem with equivalent criteria do not satisfy the decision maker, then the optimal solution is selected from some subset of points $\boldsymbol{S_1^o}, \boldsymbol{S_2^o}, \boldsymbol{S_3^o}, \boldsymbol{S_4^o}$. These subsets of Pareto points are shown in Figure 6.1 as functions $f_1(X), \ldots, f_4(X)$.

Step 2*. Choice of priority criterion of q∈**K**.*

For the choice of priority criterion on the display the message about results of the solution of λ-problem in physical and relative units is given:

Criteria (6.50) in Y^o optimum point:

$\{h_1(Y^o) = 364.0, h_2(Y^o) = 1790.7, h_3(Y^o) = \ 194.3, h_4(Y^o) = \ 47.5\}$;

$\lambda_k(Y^o), k = \overline{1,K}$- values of relative estimates (6.51):

$\{\lambda_1(Y^o) = 0.7372, \lambda_2(Y^o) = 0.6087, \lambda_3(Y^o) = 0.6087, \lambda_4(Y^o) = 0.6087\}$;

λ^o=0.6087 is the highest lower level among all relative estimates, measured in relative units:

$\lambda^o = min\ (\lambda_1(Y^o), \lambda_2(Y^o), \lambda_3(Y^o), \lambda_4(Y^o)) = \ 0.6087.$

From the theory (the Theorem 2) it is known that in an optimum point of X^o there are always two most contradictory criteria: q∈$\boldsymbol{K}$ and v∈$\boldsymbol{K}$ for which in the relative unit's precise equality is carried out:

$\lambda^o = \lambda_q(X^o) = \lambda_p(X^o), q, p \in \boldsymbol{K}, X \in S,$

and for the others it is carried out inequalities: $\lambda^o \leq \lambda_k(X^o), \forall k \in \boldsymbol{K},\ \ q \neq p \neq k$

In the model of material (6.30)-(6.35) and the corresponding λ-problem (6.42)-(6.48), such criteria are the second and third: $\lambda^o = \lambda_2(X^o) = \lambda_3(X^o)$= 0.6087, i.e. numerical symmetry is met. This symmetry will be shown in Figure 6.3, where the functions $\lambda_1(X)$ and $\lambda_3(X)$ are presented separately on the optimal point side $\boldsymbol{X^o} = \{X^o, \lambda^o\}$. For comparison, let's similarly present the functions of the most contradictory criteria $\lambda_2(X)$ and $\lambda_3(X)$ separately on the side of the optimal point $\boldsymbol{X^o} = \{X^o, \lambda^o\}$. Figures 6.3 and 6.4 show all the points and data discussed in Figure 6.2.

Typically, of this pair $\lambda^o = \lambda_2(X^o) = \lambda_3(X^o)$= 0.6087 contradictory criteria, the criterion that the decision-maker would like to improve is chosen. Such a criterion is called a "priority criterion", let's denote it $q = 3 \in \boldsymbol{K}$. This criterion is studied in conjunction with the first criterion $q = 1 \in \boldsymbol{K}$. We examine these two criteria from the set of $\boldsymbol{K} = 4$ criteria shown in Figure 6.3.

The following message is displayed on the display:

q=input ('Enter priority criterion (number) of q =') - Entered: q=3.

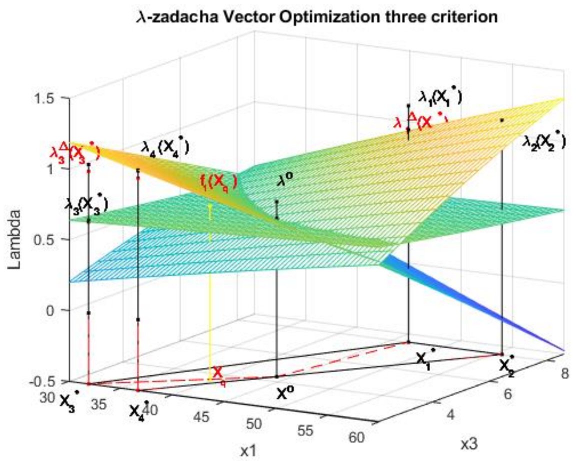

Figure 6.3. Solving λ-problems: $\lambda_2(X)$, $(\lambda_3(X)$. and $\lambda_4(X)$ in the three-dimensional coordinate system $x_1\ x_2$ and λ, $\lambda_2(X^o) = \lambda_3(X^o) = \lambda_4(X^o)$= 0.6087

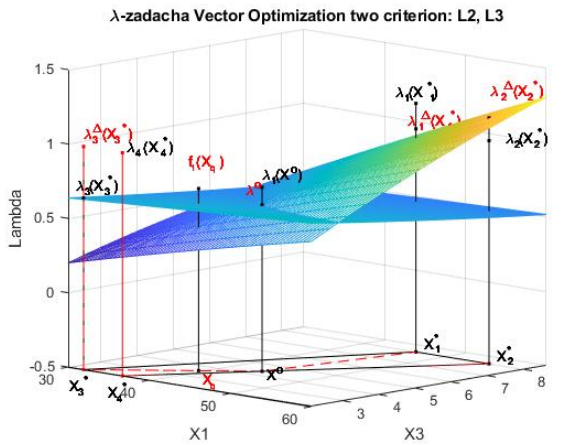

Figure 6.4. Solving λ-problems: $(\lambda_2(X)$ and $\lambda_3(X))$ in the three-dimensional coordinate system $x_1\ x_2$ λ, $\lambda_2(X^o) = \lambda_3(X^o)$= 0.6087.

Step 3. Numerical limits of change of size of a priority of criterion of *q=3*∈***K*** are defined. For priority criterion of $q = 3 \in \boldsymbol{K}$ changes of numerical limits in physical units upon transition from X^o optimum point to the point of X_q^* received on the first step at equivalent criteria are defined. *q*=3 given about criterion are given for the screen:

$$f_q(X^o) = 194.27 \leq f_q(X) \leq 210.35 = f_q(X_q^*), q \in \boldsymbol{K}. \quad (6.61)$$

In the relative units the criterion of *q*=3 changes in the following limits:

$$\lambda_q(X^o) = 0.6087 \leq \lambda_q(X) \leq 1 = \lambda_q\left(X_q^*\right), q = 3 \in \boldsymbol{K}. \quad (6.62)$$

These data it is analysed.

Step 4. *Choice of size of priority criterion* of *q*∈***K***. (Decision-making).

On the message: "Enter the size of priority criterion *fq*=" - we enter, the size of the characteristic defining structure of material: $f_q = 200$.

Step 5. *The relative assessment is calculated.*

For the chosen size of priority criterion $f_q = 200$ the relative assessment is calculated:

$\lambda q = \frac{f_q - f_q^0}{f_q^* - f_q^0} = \frac{200-194.27}{210.35-194.27} = 0.7479,$

which upon transition from X^o point to X_3^* lies in limits:

$\lambda_q(X^o) = 0.6087 \leq \lambda_q(X) \leq 0.7489 \leq \lambda_q(X_q^*), q = 3 \in \boldsymbol{K}.$

Step 6. Let's calculate coefficient of the linear approximation

Assuming the linear nature of change of criterion of $f_q(X)$ in (6.61) and according to the relative assessment of $\lambda_q(X)$, using reference methods of the linear approximation, we will calculate a constant of proportionality between $\lambda_q(X^o)$, λ_q which we will call ρ:

$$\rho = \frac{\lambda_q - \lambda_q(X^o)}{\lambda_q(X_q^*) - \lambda_q(X^o)} = \frac{0.7489-0.6087}{1-0.6087} = 0.3558, q = 3 \in \boldsymbol{K}. \quad (6.63)$$

Step 7. Let's calculate coordinates of a priority of criteria with dimension of f_q.

Assuming the linear nature of change of a vector of X^q={x_1 x_2 x_3 x_4}, *q*=3 we will determine point coordinates with dimension of *fq* =200, the relative assessment (6.53):

$$X_{\lambda=0.6596}^{q=3} = \{x_1 = X^o(1) + \rho\left(X_q^*(1) - X^o(1)\right),$$
$$x_2 = X^o(2) + \rho\left(X_q^*(2) - X^o(2)\right),$$
$$x_3 = X^o(3) + \rho\left(X_q^*(3) - X^o(3)\right),$$
$$x_4 = X^o(4) + \rho(X_q^*(4) - X^o(4))\}, \quad (6.64)$$

where $X^o = \{x_1 = 43.9, x_2 = 49.54, x_3 = 4.348, x_4 = 2.2\}$,

$X_3^* = \{x_1 = 31.9, x_2 = 59.00, x_3 = 2.1, x_4 = 7.0\}$.

As result of the decision (6.40) we will receive X^q point with coordinates: $X^q = \{x_1 = 39.63, x_2 = 52.91, x_3 = 3.54, x_4 = 3.907\}$

In the relative units the criterion of *q*=3 changes in the following limits:

$\lambda_q(X^o) = 0.546 \leq \lambda_q(X) \leq 1 = \lambda_q\left(X_q^*\right), q = 3 \in \boldsymbol{K}.$

These data it is analysed.

Step 8. Calculation of the main indexes of a point of X^q.

For the received X^q point, we will calculate: all criteria in physical units,

$f_k(X^q) = \{f_k(X^q), k = \overline{1,K}\}, f(X^q) = \{f_1(X^q) = 344.3, f_2(X^q) = 2000, f_3(X^q) = 199, f_4(X^q) = 41.7\}$

all relative estimates of criteria:

$\lambda^q = \{\lambda_k^q, k = \overline{1,K}\},\ \lambda_k(X^q) = \frac{f_k(X^q) - f_k^0}{f_k^* - f_k^0}, k = \overline{1,K},$

$\lambda_k(X^q) = \{\lambda_1(X^q) = 0.5224, \lambda_2(X^q) = 0.418, \lambda_3(X^q) = 0.7244, \lambda_4(X^q) = 0.7446\}.$

min relative estimates: $\min \lambda(X^q) = \min_{k \in K}(\lambda_k(X^q)) = 0.418.$

$P^q = [p_1^3 = 1.3868, p_2^3 = 1.7333, p_3^3 = 1.0, p_4^3 = 0.973];$

вектор приоритетов $P^q(X) = \{p_k^q = \frac{\lambda_q(X^q)}{\lambda_k(X^q)}, k = \overline{1,K}\}$:

$\lambda_k(X^q) * P^q = \{p_1^3 * \lambda_1(X^q) = 0.7244,\ p_2^3 * \lambda_2(X^q) = 0.7244,\ p_3^3 * \lambda_3(X^q) = 0.7244,$ $p_4^3 * \lambda_4(X^q) = 0.7244\}$

Min relative estimates: $\lambda^{oo} = \min(p_1^3\lambda_1(X^q), p_2^3\lambda_2(X^q), p_3^3\lambda_3(X^q), p_4^3\lambda_4(X^q)) = 0.7244$

Similarly, other Pareto points $\boldsymbol{X_t^o} = \{\lambda_t^o, X_t^o\} \in \boldsymbol{S^o}$ can be obtained.

Analysis of the results obtained. The calculated value is $f_q(X_t^o) = 199, q = 3 \in \boldsymbol{K}, q \in \boldsymbol{K}$ is usually not equal to the given $f_q = 200$. $\Delta f_q = |f_q(X_t^o) - f_q| = |199 - 200| = 1.0$ is determined by a linear approximation error: $\Delta f_{q\%} = 0.5\%$. If the error $\Delta f_q = |f_q(X_t^o) - f_q| = |199 - 200| = 1.0$, measured in physical units or as a percentage $\Delta f_{q\%} = \frac{\Delta f_q}{f_q} * 100 = 0.5\%$, greater than the specified $\Delta f, \Delta f_q > \Delta f$ then proceed to step 2, if $\Delta f_q \leq \Delta f$, then the calculation is completed.

In the process of modeling, parametric constraints (6.48) and functions can be changed, i.e., a certain set of optimal solutions is obtained. From this set of optimal solutions, we choose the final option (the decision-making process). In our example, the final option includes the parameters: $X^o = \{X^o, \lambda^o\} = \{X^o = \{x_1 = 43.9, x_2 = 49.54, x_3 = 4.348, x_4 = 2.2\}, \lambda^o = 0.6087\}$; parameters of the technical system at the given priority of the criterion:

q=3: $X^q = \{x_1 = 39.63, x_2 = 52.91, x_3 = 3.54, x_4 = 3.907\}$.

6.3. Block 3. Research, design, geometric interpretation of N-dimensional space into 2-dimensional space and selection of optimal parameters of the complex structure of the material in multidimensional mathematics

Block 3 includes 2 stages: 8 stages of research in relative units: 9 stages of research in physical units.

6.3.1. **Stage 8.** *Geometric interpretation of the results, solution in relative units when designing the structure of the material, transition from N-dimensional to two-dimensional space.*

The geometric interpretation of the results of the solution in relative units can be presented, first, by the example of the functions $\lambda_2(X), \lambda_3(X)$, secondly, separately on the example of the functions $\lambda_2(X)$ and $\lambda_3(X)$.

1. Investigation of the functions of $\boldsymbol{\lambda_2(X), \lambda_3(X)}$ to the maximum.

When studying the parameters of the material structure on the set of $\boldsymbol{S}$ points formed by constraints (6.30)-(6.35), the optimal points $X_1^*, X_2^*, X_3^*, X_4^*$, shown in Figure 6.1, are combined into a contour and represent the set of Pareto-optimal points, $\boldsymbol{S^o} \subset \boldsymbol{S}$.

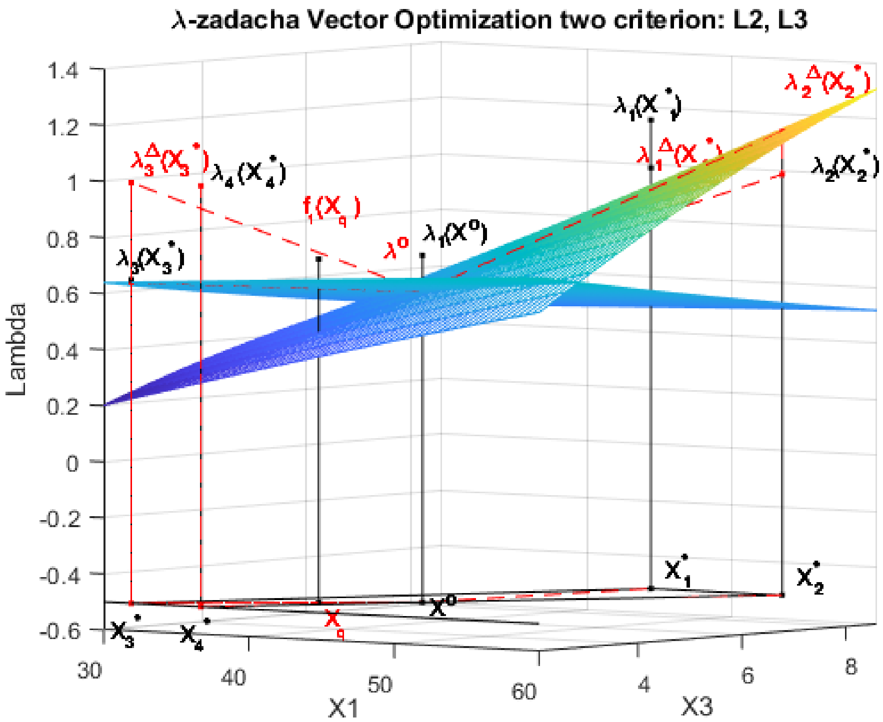

Figure 6.5. Functions $\lambda_2(X), \lambda_3(X)$ and λ^o in the λ-problem in the two-dimensional coordinate system $x_1\ x_3$ *and* λ the geometric interpretation $\lambda_2(X), \lambda_3(X)$ in a four-dimensional coordinate system of $x_1\ x_2\ x_3\ x_4$.

The coordinates of these points, as well as the characteristics of the structure of the material in relative units $\lambda_1(X), \lambda_2(X), \lambda_3(X), \lambda_4(X)$ are shown in Figure 6.2 in two-dimensional space $x_1\ x_2$ and λ, where the third axis λ is a relative estimate.

Looking at Figure 6.2, we can imagine the changes in all the functions of $\lambda_1(X), \dots, \lambda_4(X)$ in four-dimensional space $x_1, \dots, x_4$. For clarity, let's choose the two most contradictory functions $\lambda_2(X), \lambda_3(X)$, shown in Figure 6.3, and represent these functions $\lambda_2(X), \lambda_3(X)$ in Figure 6.5.

Let's consider Figure 6.5 optimal point, X_3^*. Function $\lambda_3(X)$ – in relative units, formed from the function $f_3(X)$ – in physical units with variable coordinates $\{x_1\ x_3\}$ and with constant coordinates $\{x_2 = 1790.7, x_4 = 47.5\}$, taken from the optimal point X^o (6.50). At the point X_3^* the relative estimate is $\lambda_3(X_3^*) = 0.6441$ – shown in Figure 6.2 with a black dot.

But we know that the relative estimate of $\lambda_3(X_3^*)$ obtained from the function $f_3(X_3^*)$ in the third step, it is equal to one, let's denote it as $\lambda_3^\Delta(X_3^*) = 1$ – shown in Figure 6.5 with a red dot. The difference between $\lambda_3^\Delta(X_3^*) = 1$ and $\lambda_3(X_3^*) = 0.6441$ is an error Δ= 0.3559 of the transition from a four-dimensional (and in the general case *N*-dimensional) to a two-dimensional region. Let us connect the relative estimates λ^o and $\lambda_3^\Delta(X_3^*)$, lying between the points $X^o and\ X_3^*$.

Similarly, X_3^* let's imagine the point X_2^* with corresponding relative estimates of $\lambda_2(X_2^*)$= 1.1530 in $\{x_2\ x_3\}$ coordinates and $\lambda_2^\Delta(X_1^*) = 1$ obtained in the coordinates $\{x_1\ x_2\ x_3\ x_4\}$. A linear function connecting the points λ^o and $\lambda_2^\Delta(X_2^*)$ in relative units, it characterizes the function $f_2(X)$ in relative units in the four-dimensional dimension of the parameters $x_1, \dots, x_4$.

And in general, segments are $\lambda_2^\Delta(X_2^*)$ -λ^o- $\lambda_3^\Delta(X_3^*)$ represent the geometric interpolation of the functions $f_2(X)$ and $f_3(X)$ in relative units in the four-dimensional dimension of the parameters $x_1, \dots, x_4$.

2. Investigation of the functions of $\lambda_2(X), \lambda_3(X)$ separately into the maximum and minimum of the four-dimensional system.

Let's conduct a study of the functions $f_2(X)$, represented in relative units: , $\lambda_2(X)$, which is shown in Figure 6.6.

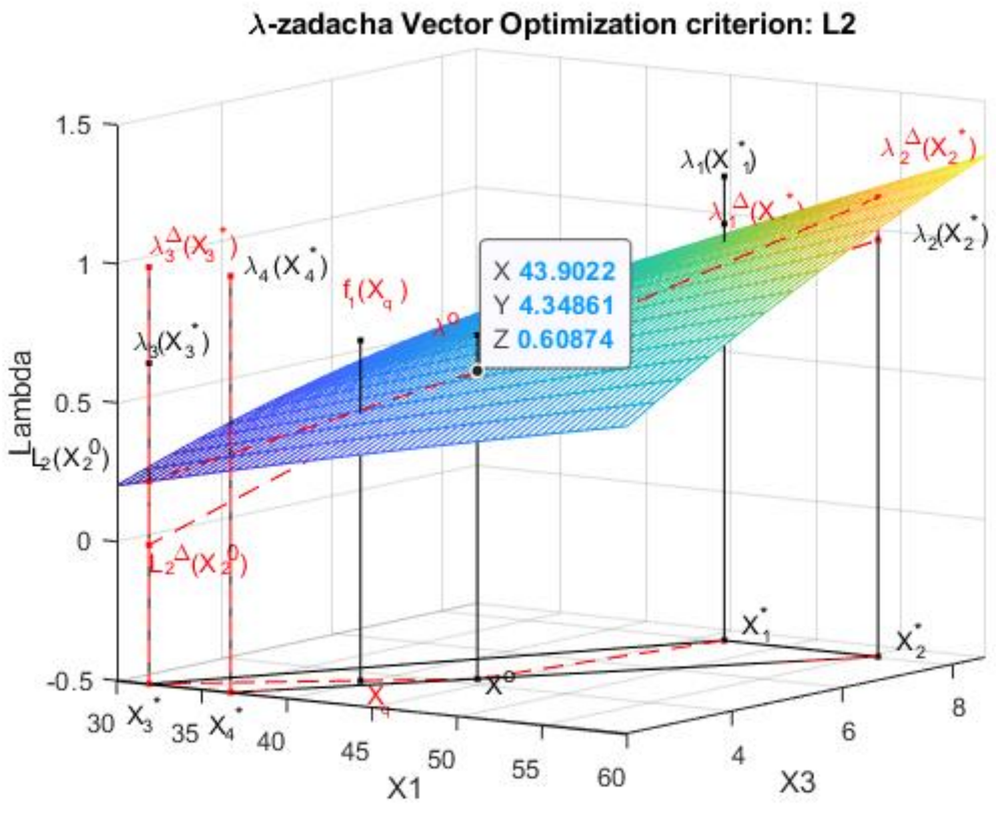

Figure 6.6. Functions $\lambda_2(X)$ and λ^o in the λ-problem in the two-dimensional coordinate system $x_1\ x_3$ and λ, the geometric interpretation $\lambda_2(X)$ in a four-dimensional coordinate system $x_1\ x_2\ x_3\ x_4$ (highlighted in red).

To estimate the maximum $\lambda_2^{\Delta}(X_2^*) = 1$ values of the second criterion in relative units in the four-dimensional coordinate system $\{x_1\ x_2\ x_3\ x_4)$, (highlighted in red) the data obtained in the third step (6.36) of the matrix λ(X^*) are used. To estimate the minimum $\lambda_2^{\Delta}(X_2^0) = 0$ values of the second criterion in relative units in the four-dimensional coordinate system $\{x_1\ x_2\ x_3\ x_4\}$, (highlighted in red) the data obtained in the third step are used the data obtained in the third step λ(X^0).

The maximum difference between $\lambda_2^{\Delta}(X_2^*) = 1$ (four-dimensional system) and $\lambda_2(X_1^*) = 1.1530$ (two-dimensional system) is an error Δ=0.1530 of the transition from a four-dimensional (and in the general case N-dimensional) to a two-dimensional region.

The minimum difference between $\lambda_2^{\Delta}(X_2^0) = 0$ (four-dimensional system) and $\lambda_2(X_2^0) = 0.2290$ (two-dimensional system) is an error Δ = 0.2290 of the transition from a four-dimensional (and in the general case *N*-dimensional) to a two-dimensional region.

We will connect the relative estimates with a linear function: $\lambda_2^{\Delta}(X_2^*)\ \lambda^o$ and $\lambda_2^{\Delta}(X_2^0)$, which lie between the points: $X_2^*\ X^o$ and X_2^0. In general, the linear segments are $\lambda_2^{\Delta}(X_2^*)$ -λ^o- $\lambda_2^{\Delta}(X_2^0)$ represent the geometric interpolation of the function $f_2(X)$ in relative units $\lambda_2(X)$ in the four-dimensional dimension of the parameters $x_1, \dots, x_4$.

Let's study the function $f_3(X)$, represented in relative units: $\lambda_3(X)$, shown in Figure 6.7.

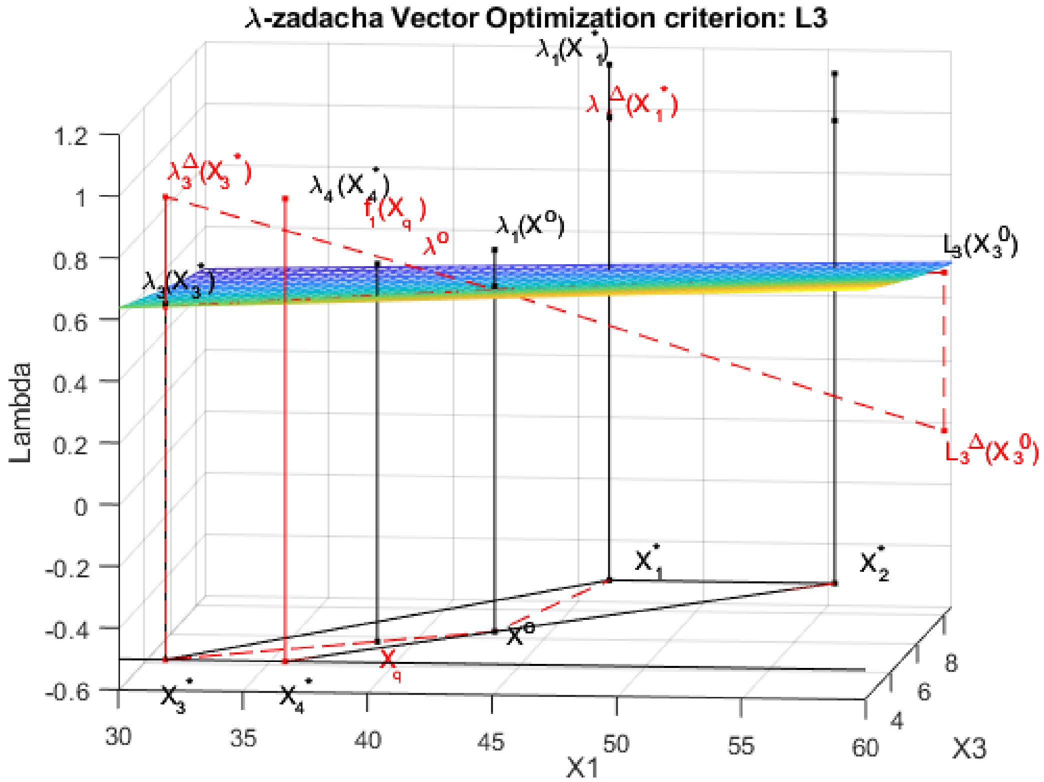

Figure 6.7. Functions, $\lambda_3(X)$ and λ^o in the λ-problem in the two-dimensional coordinate system $x_1\ x_3$ and λ; and the geometric interpretation $\lambda_3(X)$ in a four-dimensional coordinate system, (highlighted in red).

To estimate the maximum $\lambda_3^{\Delta}(X_3^*) = 1$ values of the third criterion in relative units in the four-dimensional coordinate system $\{x_1\ x_2\ x_3\ x_4\}$ (highlighted in red) the data obtained in the third step (6.36) of the matrix λ(X^*) are used. To estimate the minimum $\lambda_3^{\Delta}(X_0^0) = 0$ values of the first criterion in relative units in the four-dimensional coordinate system $\{x_1\ x_2\ x_3\ x_4\}$, (highlighted in red), the data obtained in the third step (6.36) of the matrix λ(X^0) are used.

The maximum difference is between $\lambda_3^{\Delta}(X_3^*) = 1$ (four-dimensional system) and $\lambda_3(X_3^*) = 0.6441$ (two-dimensional system) is the error Δ=0.3559 of the transition from a four-dimensional (and in the general case *N*-dimensional) to a two-dimensional region.

The minimum difference between $\lambda_3^{\Delta}(X_3^0) = 0$ (four-dimensional system) and $\lambda_3(X_3^0) = 0.5629$ (two-dimensional system) is an error Δ=0.4371 of the transition from a four-dimensional (and in the general case *N*-dimensional) to a two-dimensional region.

Let us connect the relative estimates by a linear function $\lambda_3^{\Delta}(X_3^*)\ \lambda^o$ and $\lambda_3^{\Delta}(X_3^0)$, lying between the points $X_3^*\ X^o$ and X_3^0. And in general, the segments are $\lambda_3^{\Delta}(X_3^*)$ -λ^o- $\lambda_3^{\Delta}(X_3^0)$ represent the geometric interpolation of the functions $f_3(X)$ in relative units $\lambda_3(X)$ in the four-dimensional dimension of the parameters$x_1, \dots, x_4$.

Thus, for the first time in domestic and foreign practice, the transition and its geometric interpretation from the *N*-dimensional to the two-dimensional dimension of the function in vector problems of mathematical programming with the corresponding approximation errors is shown.

6.3.2. **Stage 9.** *Geometric interpretation of the results of the solution of the VPMP – a model of the structure of the material when designing in a three-dimensional coordinate system* ***in physical units****.*

At the fifth step of the algorithm, we calculated the parameters of the optimum point with equivalent criteria: $\boldsymbol{X^o} = \{X^o, \lambda^o\} = \{Y^o = \{x_1 = 43.9, x_2 = 49.54, x_3 = 4.348, x_4 = 2.2\}, \lambda^o = 0.6087\}$, in the two-dimensional coordinate system x_1, x_2 Figure 6.1 and in the three-dimensional coordinate system x_1, x_2 and λ. in relative units in Figures 6.2, 6.3, 6.4 when designing.

Figure 6.5 shows: optimum points X_1^*, X_3^*, with corresponding relative estimates $\lambda_1(X_1^*)\ \lambda_1^{\Delta}(X_1^*)$, $\lambda_3(X_3^*)\ \lambda_3^{\Delta}(X_3^*)$ and linear functions $\lambda^o \lambda_1^{\Delta}(X_1^*)$, $\lambda^o \lambda_3^{\Delta}(X_3^*)$ in relative units, which characterize the functions $f_1(X)$, $f_3(X)$ in the four-dimensional dimension of the parameters $x_1, \dots, x_4$..

Let us examine and present these parameters for each characteristic of the structure of the material (criterion): $f_1(X), f_2(X), f_3(X), f_4(X)$ ***in physical units***.

Stage 9.1. Geometric interpretation of the results of the VPMP solution – the first characteristic of the structure of the material in the design in physical units.

The first characteristic of the structure material $f_1(X)$ is formed in 6.1.4:

$$\max h_1(X) \equiv 323.84 - 2.25y_1 - 3.49y_2 + 10.72y_3 + 13.124y_4 + 0.0968y_1y_2 - 0.062y_1y_3 - 0.169y_1y_4 + 0.0743y_2y_3 - 0.1y_2y_4 - 0.0036y_3y_4 + 0.0143y_1^2 + 0.0118y_2^2 - 0.2434y_3^2 - 0.5026y_4^2, \quad (6.30)$$

Let us present a geometric interpretation of the function $h_1(Y)$ in physical units with variable coordinates $\{y_1\ y_3\}$ and with constant coordinates $\{y_2 = \{49.54, y_4 = 2.2\}$, taken from the optimal point Y^o (6.49) in Figure 6.8.

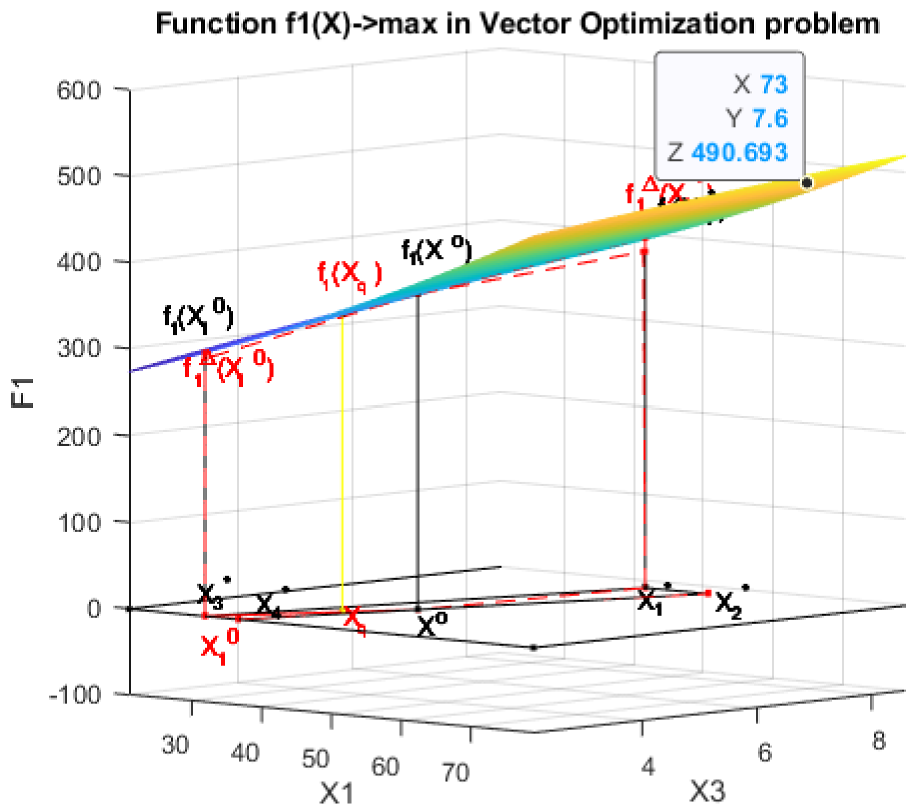

Figure 6.8. Function $f_1(X)$ in a two-dimensional coordinate system $x_1\ x_3$ and a geometric interpretation of the function $f_1(X)$ in the coordinate system $x_1\ x_2\ x_3\ x_4$.

The coordinates of ***the maximum*** point are $Y_1^* = \{y_1 = 46.7636, y_3 = 8.0\}$ (denoted as X1omax in Figure 6.8). The value of the objective function is h_1^*=FX1omax = 387.9.

The coordinates of ***the minimum*** point are $X_1^0 = \{x_1 = 31.9, x_3 = 2.1\}$ (denoted as X1omin in Figure 6.8). The value of the objective function is F_1^0=FX1omin = 296.6.

The coordinates of the point are $X^o = \{x_1 = 43.9, x_3 = 4.348\}$ (in Figure 6.8 it is indicated as X1o0). Value of the objective function $f_1(X^o)$ =FX1o0 = 364.0.

Optimum points: X_1^* with four parameters calculated in step 1 and the criterion value is $f_1^* = f_1(X_1^*) = -387.9$; point X_1^0 with the value of the criterion $f_1^0 = f_1(X_1^0) = 296.6$ – In the figure, they are denoted $(f_1^\Delta(X_1^*), f_1^\Delta(X_1^0)$.

A linear function connecting the points $f_1(X^o)$ and $f_1^\Delta(X_1^*)$ in physical units, it characterizes the function $f_1(X)$ in the four-dimensional dimension of the parameters $x_1, \dots, x_4$. And in general, the segments are $f_1^\Delta(X_1^*)$ - $f_1(X^o)$ - $f_1^\Delta(X_1^0$ represent the geometric interpolation of the function $f_1(X)$ in the four-dimensional dimension of the parameters $x_1, \dots, x_4$.

Stage 9.2. Geometric interpretation of the results of the VPMP solution – the second characteristic of the structure material in the design in physical units.

The second characteristic of the structure material $f_2(X)$ is presented in 6.1.4:

$$min\ f_2(X) \equiv 795.72 +23.89x_1 +30.866x_2 -25.8586x_3 -45.0026x_4 -0.7683x_1x_2 +0.4703x_1x_3 +0.7472x_1x_4 -0.1283x_2x_3 +0.3266x_2x_4 -0.0112x_3x_4 +0.0398x_1^2 +0.0365x_2^2 +3.2x_3^2 +2.6457x_4^2, \quad (6.32)$$

The coordinates of ***the maximum*** point are $X_2^* = \{x_1 = 55.6, x_3 = 8.0\}$ (denoted as X2omax in Figure 6.9). The value of the objective function is F_2^*=FX2omax = 1361.4.

The coordinates of ***the minimum*** point are $X_2^0 = \{x_1 = 31.9, x_3 = 2.1\}$ (denoted as X2omin in Figure 6.9). The value of the objective function is F_2^0=FX2omin = 2458.5.

The coordinates of the point are $X^o = \{x_1 = 43.9, x_3 = 4.348\}$ (in Figure 6.9 it is indicated as X2o0). The value of the objective function is $f_2(X^o)$ =FX2o0 = 1790.7.

Optimum points: X_2^* with four parameters calculated in step 1 and the value of the criterion is $f_2^* = f_2(X_2^*) = 1361.4$; point X_2^0 with the value of the criterion $f_2^0 = f_2(X_2^0) = -2458.5$ – (in Figure 6.9 it is indicated as $f_2^\Delta(X_2^*)$, $f_2^\Delta(X_2^0)$).

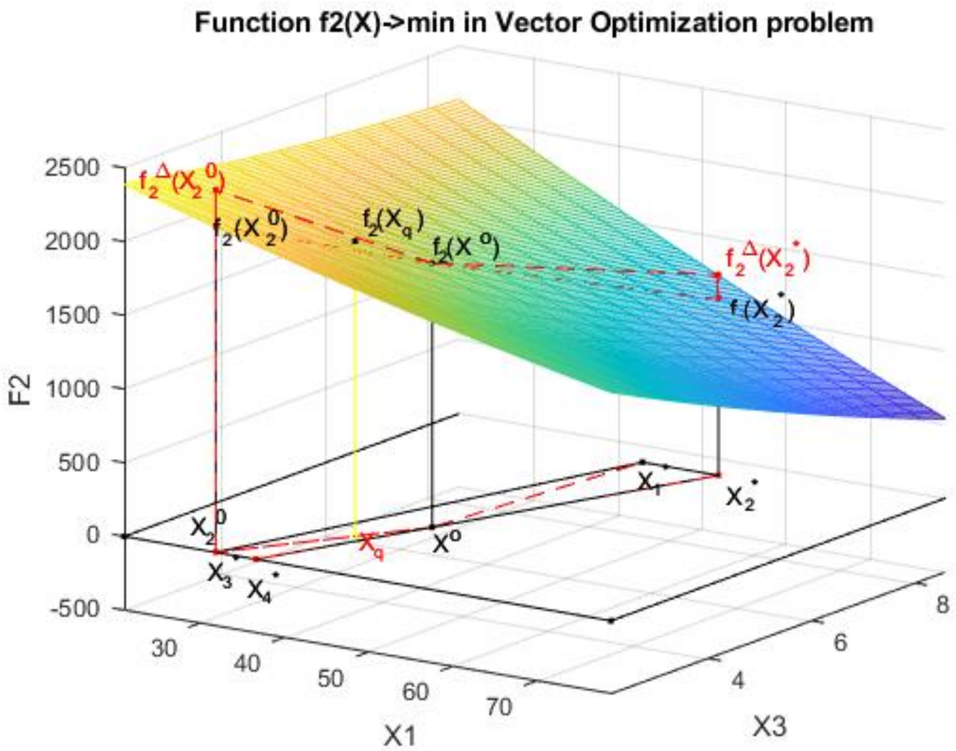

Figure 6.9. Function $f_2(X)$ in a two-dimensional coordinate system $f_2(X)$ and a geometric interpretation of the function $f_2(X)$ in the coordinate system $x_1\ x_2\ x_3\ x_4$.

A linear function connecting the points $f_2(X^o)$ and $f_2^{\Delta}(X_2^*)$ in physical units, it characterizes the function $f_2(X)$ in the four-dimensional dimension of the parameters $x_1, \dots, x_4$. And in general, the segments are $f_2^{\Delta}(X_2^*)$ - $f_2(X^o)$ - $f_2^{\Delta}(X_2^0)$ represent the geometric interpolation of the function $f_2(X)$ in the four-dimensional dimension of the parameters $x_1, \dots, x_4$.

Stage 9.3. Geometric interpretation of the results of the solution of the VPMP – ***the third characteristic*** of the structure of the material in the design in physical units.

The third characteristic of the structure of the material $f_3(X)$ is formed in 6.1.4:

$max\ h_3(Y) \equiv 110.22 + 0.7918y_1 + 1.73y_2 - 0.3713y_3 - 2.20y_4 - 0.0132y_1y_2 - 0.008y_1y_3 + 0.0193y_1y_4 - 0.0172y_2y_3 + 0.0161y_2y_4 - 0.0006y_3y_4 - 0.0004y_1^2 - 0.0002y_2^2 + 0.0335y_3^2 + 0.124y_4^2\},$ (6.31)

Let us present a geometric interpretation of the function $f_3(X)$ in physical units with variable coordinates $\{x_1\ x_3\}$ and with constant coordinates $y_2 = \{49.54, y_4 = 2.2\}$, taken from the optimal point Y^o (6.49) in Figure 6.10.

The coordinates of ***the maximum*** point are $X_3^* = \{x_1 = 31.9, x_3 = 2.1\}$ (in Figure 6.10 denoted as X3omax). The value of the objective function is F_3^*=FX3omax = 210.3.

The coordinates of ***the minimum*** point are $X_3^0 = \{x_1 = 78.16, x_3 = 8.0\}$ (denoted as X3omin in Figure 6.10). The value of the objective function is f_3^0=FX3omin = 169.26.

The coordinates of the point are $X^o = \{x_1 = 43.9, x_3 = 4.348\}$ (in Figure 6.10 it is indicated as X3o0). The value of the objective function is $f_3(X^o)$ =FX3o0 = 194.3.

Optimum points: X_3^* with four parameters calculated in step 1 and the criterion value is $f_3^* = f_3(X_3^*) = -210.3$; point X_3^0 with the value of the criterion $f_3^0 = f_3(X_3^0) = 169.26$ – (in Figure 6.8 denoted as $f_3^{\Delta}(X_3^*)$, $f_3^{\Delta}(X_3^0)$).

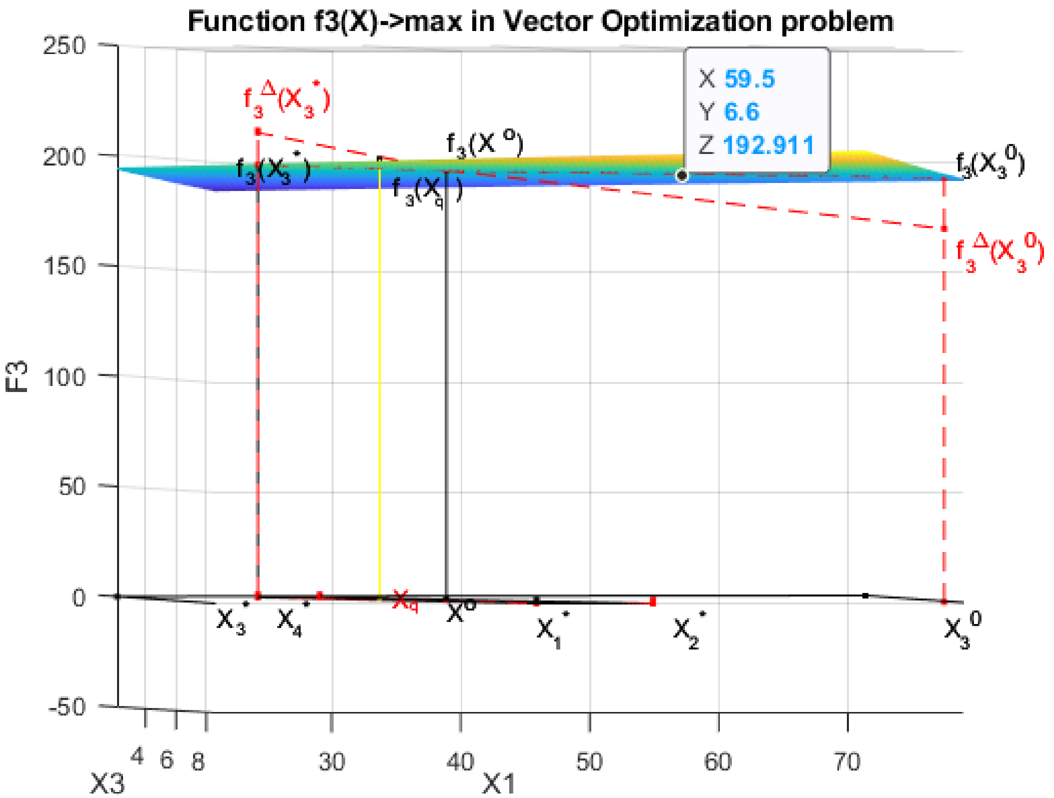

Figure 6.10. Function $f_3(X)$ in a two-dimensional coordinate system $x_1\ x_3$ and a geometric interpretation of the function $f_3(X)$ in the coordinate system $x_1\ x_2\ x_3\ x_4$.

A linear function connecting the points $f_3(X^o)$ and $f_3^{\Delta}(X_3^*)$ in physical units, it characterizes the function $f_3(X)$ in the four-dimensional dimension of the parameters $x_1\ x_2\ x_3\ x_4$.

And in general, the segments are $f_3^{\Delta}(X_1^*)$ - $f_3(X^o)$ - $f_3^{\Delta}(X_3^0)$ represent the geometric interpolation of the function$f_3(X)$ in the four-dimensional dimension of the parameters $x_1\ x_2\ x_3\ x_4$.

Stage 9.4. Geometric interpretation of the results of the solution of the VPMP – ***the fourth characteristic*** of the structure of the material in the design in physical units.

The fourth characteristic of the structure of the material $f_4(X)$. is presented in 6.1.4:

$max\ h_4(Y)$≡$21.004 - 0.0097y_1 - 0.841y_2 - 0.4326y_3 + 1.1723y_4 + 0166y_1y_2 + 0.085y_1y_3 - 0.0001y_1y_4 + 0.0523y_2y_3\ + 0.0002y_2y_4 + 0.0006y_3y_4 - 0.0022y_1^2 + 0.0035y_2^2 + 0.006y_3^2 - 0.0311y_4^2\}\}$, (6.33)

Let us present a geometric interpretation of the function $f_4(X)$ in physical units with variable coordinates $\{x_1\ x_3\}$ and with constant coordinates $y_2 = \{49.54, y_4 = 2.2\}$, taken from the optimal point Y^o (6.49) in Figure 6.11.

The coordinates of ***the maximum*** point are $X_4^* = \{x_1 = 36.70, x_3 = 2.1\}$ (in Figure 6.11 denoted as X4omax). The value of the objective function is F_4^*=FX4omax = 30.714.

The coordinates of ***the minimum*** point are $X_4^0 = \{x_1 = 62.71, x_3 = 8\}$ (denoted as X4omin in Figure 6.11). The value of the objective function is F_4^*=FX4omin = -73.62.

The coordinates of the point are $X^o = \{x_1 = 43.9, x_3 = 4.348\}$ (in Figure 6.11 it is indicated as X4o0). The value of the objective function is $f_4(X^o)$ - FX4o0 = 47.5.

Optimum points: X_4^* with four parameters calculated in step 1 and the value of the criterion is $f_4^* = f_4(X_4^*)\ = 30.714$; point X_4^0 with the value of the criterion $f_4^0 = f_4(X_4^0) = -73.62$ – The figure shows $(f_4^\Delta(X_4^*), f_4^\Delta(X_4^0)$.

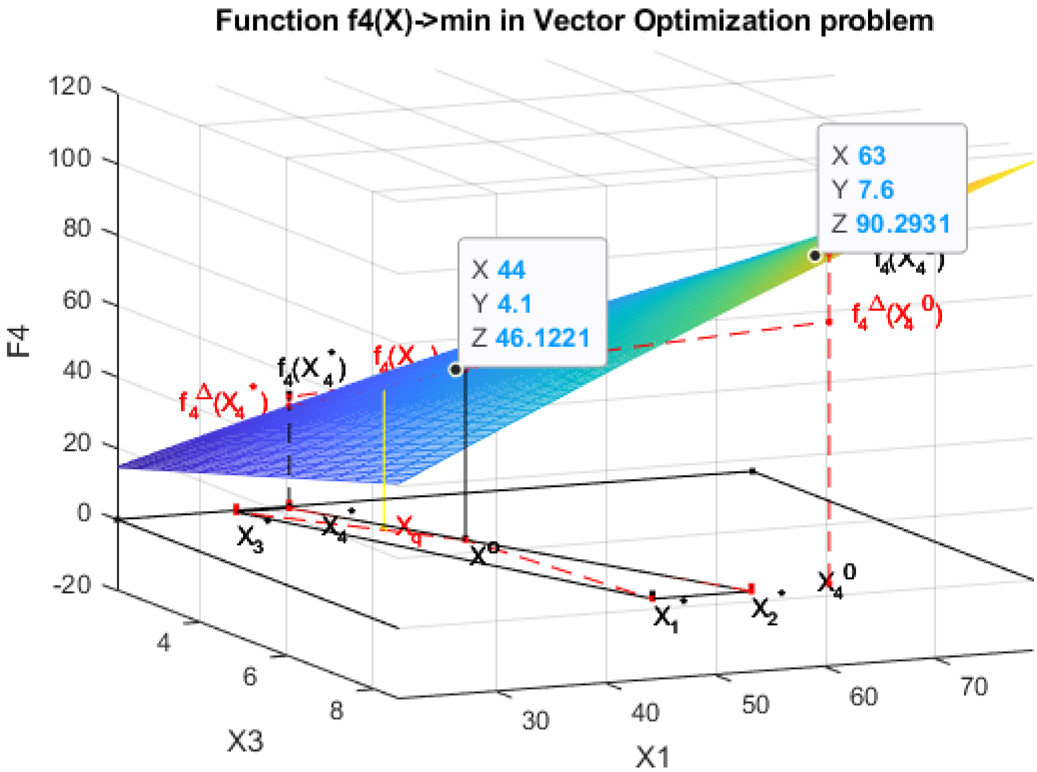

Figure 6.11. Function $f_4(X)$ in a two-dimensional coordinate system $x_1\ x_3$ and a geometric interpretation of the function $f_4(X)$ in the coordinate system $x_1\ x_2\ x_3\ x_4$.

A linear function connecting the points $f_4(X^o)$ and $f_4^\Delta(X_4^*)$ in physical units, it characterizes the function $f_4(X)$ in the four-dimensional dimension of the parameters $x_1, \dots, x_4$.

And in general, the segments are $f_4^\Delta(X_4^*)$ - $f_4(X^o)$ - $f_4^\Delta(X_4^0)$ represent the geometric interpolation of the function $f_4(X)$ in the four-dimensional dimension of the parameters $x_1, \dots, x_4$.

Taken together, the software version gives the following results: optimum point - X^o;

characteristics (criteria) – $F(X^o) = \{f_1(X^o), f_2(X^o), f_3(X^o), f_4(X^o)\}$;

relative evaluations – $\lambda(X^o) = \{\lambda_1(X^o), \lambda_2(X^o), \lambda_3(X^o), \lambda_4(X^o)\}$;

the maximum relative estimate λ^o, such that $\lambda^o \leq\ \lambda_k(X^q), k = \overline{1, K}$.

the optimum point with the priority of the qth criterion is X^q;

characteristics (criteria) – $F(X^q) = \{f_1(X^q), f_2(X^q), f_3(X^q), f_4(X^q)\}$;

relative evaluations – $\lambda(X^o) = \{\lambda_1(X^q), \lambda_2(X^q), \lambda_3(X^q), \lambda_4(X^q)\}$;

the maximum relative estimate λ^{oo}, such that $\lambda^{oo} \le p_k^q \lambda_k(X^q), k = \overline{1,K}$.

Conclusion on the section. The problem of developing mathematical methods of vector optimization and making an optimal decision based on them in a complex structure of the material based on a certain set of experimental data and functional characteristics is one of the most important tasks of system analysis and design. The paper develops a methodology for design automation by: building a mathematical model of the material under conditions of certainty and uncertainty; development of methods for solving a vector problem and selection of optimal material parameters for a variety of characteristics.

7. Comparison of applied methods of multidimensional mathematics with methods of artificial intelligence.

Let us evaluate the applied methods of multidimensional mathematics - {axiomatics of Mashunin Yu.K., principles of optimality and methods for solving vector problems of mathematical (convex) programming}, presented in the third and fourth sections of this work, and compare them with the methods of artificial intelligence. Using the theory of vector optimization, we obtained for an engineering system (in particular, a technical system, the structure of a material):

optimum point - $X^o = (x_j^o, j = \overline{1,N}$};

characteristics (criteria) – $F(X^o) = \{f_k(X^o), k = \overline{1,K}\}$;

relative evaluations – $\lambda(X^o) = \{\lambda_k(X^o), k = \overline{1,K}\}$, which lie within $\{0 \le \lambda_k(X^o) \le 1$ (100%), $k = \overline{1,K}\}$, and is easily translated into physical data.

Can these results be obtained by artificial intelligence, which usually functions on the principle of brute force? The answer is, "No." Artificial intelligence can only get an approximate result that a person has set, but why this result is better than other results should also be evaluated by a person based on intuition.

Thus, the developed theory of vector optimization can be the mathematical apparatus of computational intelligence of artificial intelligence.

Conclusions

The problem of developing mathematical methods of multidimensional mathematics in application to the vector problem of optimization and making an optimal decision on their basis of the structure of the material based on a certain set of functional characteristics and experimental data is one of the most important tasks of system analysis and design of the structure of the material.

The paper develops a methodology for design automation by: building a mathematical model of an engineering system under conditions of certainty and uncertainty; development of methods for solving a vector problem. The construction of a mathematical and numerical model for the selection of optimal parameters of a complex technical system and material of a complex structure and their implementation by a variety of characteristics is presented

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